BULLETIN

OF THE

AMERICAN

MATHEMATICAL SOCIETY

A Historical and Critical Review
of Mathematical Science

EDITED BY

EARLE RAYMOND HEDRICK
ARNOLD DRESDEN WILLIAM RAYMOND LONGLEY

WITH THE ASSISTANCE OF

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J. H. M. WEDDERBURN

Volume XXXIII, Number 1

January-February, 1927

PUBLISHED BY THE SOCIETY
MENASHA, WIS., AND NEW YORK

1927

Published Bimonthly

Whole No. 346 $7.00 a Year

Entered as a second class matter at the post office at Menasha, Wis., May 13, 1926.

JOURNALS OF THIS SOCIETY

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PARTICIPATION IN THE PUBLICATION OF THE AMERICAN JOURNAL OF MATHEMATICS

An event which may prove to be of the highest importance is the consummation of the agreement entered into by the American Mathematical Society and the Johns Hopkins University, by which, beginning with 1927, they share in the conduct of the American Journal of Mathematics. The Society will participate in the editorial responsibility for the Journal and will contribute very substantially to its financial support. The size will be practically doubled.

The editorial board for 1927 will consist of Professors G. D. Birkhoff, A. B. Coble, Abraham Cohen, G. C. Evans, and Frank Morley (chairman). It has been agreed that the character of the Journal shall continue as in the past, the editors being given a free hand to develop the publication along traditional lines. Members of the Society are invited to contribute of their important papers. With the renewed vigor possible under the cooperation arranged, the Journal and Transactions should stand together in the front rank of the world's mathematical periodicals.

With the publication of the Colloquium Lectures, the Bulletin, and the Transactions, and the new venture in supporting the Journal, the Society expects through its efforts to provide for some twenty-two hundred pages annually. This is possible only because of the Endowment Fund, the generous support of the Sustaining Members, and the splendid assistance of the General Education Board, which has entrusted the National Academy with a sum of money for the support of scientific publication.

R. G. D. RICHARDSON,
Secretary.

THE FIFTIETH REGULAR MEETING OF THE SAN FRANCISCO SECTION

The fiftieth regular meeting of the San Francisco Section of the Society was held at the University of California on Saturday, October 30, 1926. Professor Cajori acted as temporary chairman till the arrival of the chairman, Professor Blichfeldt. The total attendance was forty, including the following twenty-two members of the Society:

Alderton, Allardice, A. B. D. Andrews, Bernstein, Blichfeldt, Buck, Cajori, Corbin, R. L. Green, M. W. Haskell, E. R. Hedrick, Hotelling, Frank Irwin, D. H. Lehmer, D. N. Lehmer, Sophia Levy, W. A. Manning, J. H. McDonald, F. R. Morris, Pauline Sperry, A. R. Williams, Wong.

The following officers were elected for the year: Chairman, Professor R. E. Allardice; Secretary, Professor B. A. Bernstein; Program Committee, Professors E. T. Bell, Daniel Buchanan, Frank Irwin, B. A. Bernstein (ex officio).

The Section accepted the invitation of President Klinck of the University of British Columbia that the 1927 Summer meeting be held in June at the University of British Columbia. It was decided to hold the next fall meeting on October 29, 1927, at the University of California.

Titles and abstracts of papers read at the meeting follow. Professor McDonald's paper was delivered at the request of the Program Committee. The papers of Professors Bell and Wear were read by title.

1. Professor J. H. McDonald: *The relations between the algebraic and transcendental solution of certain problems.*

The problems referred to belong to the integral calculus. The determination of integrals requires us to decide whether two elliptic integrals are transformable into each other, whether a given radical admits an Abel-pseudo-elliptic integral, whether a given algebraic relation has an associated elliptic integral. These problems admit enunciations which are arithmetic, algebraic, or function-theoretic. It is the object of the colloquium to consider the different aspects of these problems and to present contributions to their solution.

2. Professor E. T. Bell: *Reduction formulas for the number of representations of integers in certain quadratic forms.*

The algebraic method of the paper enables us to pass from formulas for numbers of representations in a quadratic form in $r \geqq 2$ indeterminates to the like for rs indeterminates, $s > 1$. It is applied to obtain complete proofs of the results stated by Liouville (Journal de Mathématiques, vol. 10, pp. 43–49; vol. 11, p. 211).

3. Professor E. T. Bell: *A diophantine automorphism.*

This paper appears in full in the present issue of this Bulletin.

4. Professor B. A. Bernstein: *Note on the dual of a boolean expression.*

The author obtains a new rule for writing down the dual of an expression in boolean algebras, and he makes some application of this rule.

5. Professor Florian Cajori: *Frederick the Great on mathematics and mathematicians.*

The author describes Frederick the Great's relations to Euler, Maupertuis, Lagrange, D'Alembert and Lambert, and points out that, despite his total lack of appreciation of mathematics, Frederick gave a great stimulus to the progress of this science through the patronage extended to mathematicians at the Berlin academy.

6. Professor Florian Cajori: *Madame du Châtelet on fluxions.*

The author gives Madame du Châtelet's interpretation of Newton's lemmas in the *Principia* relating to the concept of a limit.

7. Professor Florian Cajori: *Circuelo on the names "arithmetical" and "geometrical" proportion and progression.*

The author describes and comments on Circuelo's explanation of the choice of the adjectives "arithmetical" and "geometrical" in naming proportions and progressions.

8. Professor E. R. Hedrick: *A necessary and sufficient condition for the Borel theorem in a general type of space.*

In this paper it is shown that a necessary and sufficient condition for the validity of the Borel theorem in very general types of spaces is that if any family F of closed sets has no point in common, then a finite number of sets of F exist which have no point in common. Another form of this statement is useful. If in any family F of closed sets, every finite subfamily has a point in common, the entire family has a point in common. The relation of the Borel theorem to other fundamental ideas, such as the Dedekind cut, which is well known, is thrown into clearer light through the present formulation, which is simpler and more symmetric, in some respects, than the usual statements of the Borel theorem.

9. Dr. Harold Hotelling: *An application of analysis situs to statistics.*

The frequency distribution of a correlation coefficient between two sets of numbers derived by manipulation from a common body of data is important in some kinds of statistical work. These frequency distributions have however been studied only experimentally, if at all, and little is known about them. The author shows that their exact determination would be equivalent to solving certain metrical problems concerning special transformations of a hypersphere into itself. J. W. Alexander's theorems on invariant points reveal properties of the distributions.

10. Professor D. N. Lehmer: *Note on a theorem on factorization of numbers.*

This paper appears in full in the present issue of this Bulletin.

11. Professor L. E. Wear: *A variation in the idea of a line integral.*

Under suitable conditions the line integral $\int_L f(x, y)dx$ is $\lim_{n\to\infty} \sum_{i=1}^{n} f(\xi_i, \eta_i)\Delta x_i$, where n is the number of intervals into which the arc of the path has been divided and ξ_i, η_i is a point on the path within the ith interval. The perpendiculars to Ox from the ends of the intervals intercept the lengths Δx_i on the axis of x; they intercept arcs Δs_i on some suitable curve, C. The integral $\int_L f(x, y)ds$ will be defined as $\lim_{n\to\infty} \sum_{i=1}^{n} f(\xi_i, \eta_i)\Delta s_i$, where the values $f(\xi_i, \eta_i)$ are to be taken on the path L, and Δs_i are arcs of the curve C. By using the equations of L and C the integral is reduced to an ordinary definite integral. The ideas can be extended to the general case of $\int_L Mdx+Ndy$ and the analog of Green's Theorem can be written down. Similarly for higher spaces.

12. Professor W. A. Manning: *On simply transitive primitive groups.*

Three of the theorems in Rietz's memoir on *Primitive groups of odd order* (American Journal, vol. 26 (1904), p. 1) have to do with simply transitive primitive groups in which all the transitive constituents of a subgroup leaving one letter fixed are primitive. The present author shows that they are all special cases of the following theorem: *If all the transitive constituents of G_1 (the subgroup that leaves fixed one letter of a simply transitive primitive permutation-group) are primitive, G_1 is a simple isomorphism between its transitive constituents.* The author also proves the following theorem: *If one and only one of the transitive constituents of G_1 (defined above) is an imprimitive group, the order of G_1 is the same as that of its imprimitive constituent.*

B. A. BERNSTEIN,
Secretary of the Section.

THE OCTOBER MEETING IN NEW YORK

The two hundred fifty-first regular meeting of the Society was held at Columbia University, on Saturday, October 30, 1926, extending through the usual morning and afternoon sessions. The attendance included the following fifty-five members of the Society:

Alexander, W. L. Ayres, A. A. Bennett, Birkhoff, Bower, B. H. Camp, Carrié, Abraham Cohen, Doermann, Eisenhart, Fiske, Fite, Fort, Gehman, Gronwall, Hedlund, Himwich, Huber, Dunham Jackson, Joffe, R. A. Johnson, O. D. Kellogg, A. L. Kimball, B. F. Kimball, Koopman, Kormes, Lamson, Langman, Lefschetz, Littauer, Michal, Frank Morley, Richard Morris, Mullins, Pepper, Pfeiffer, Pierpont, Raudenbush, Raynor, Reddick, R. G. D. Richardson, Ritt, Seely, Siceloff, Smail, Virgil Snyder, M. H. Stone, Teach, T. Y. Thomas, Veblen, Wedderburn, Weida, Anna Pell Wheeler, Whittemore, W. A. Wilson.

The Secretary announced the election of the following eighteen persons to membership in the Society:

Miss Julia Wells Bower, Vassar College;
Mr. Edward John Braun, Wayland Academy;
Professor Loren G. Butler, Albany College;
Dr. Ralph Douglas Doner, Purdue University;
Mr. Herbert Pulse Evans, University of Wisconsin;
Mr. Joel S. Georges, University of Chicago;
Professor Roy French Graesser, University of Arizona;
Mr. Joshua V. Longenecker, Farmers and Bankers Life Insurance Company, Wichita;
Mr. Marcus Evans Mullings, University of Cincinnati;
Mr. Louis John Paradiso, Lehigh University;
Mr. Herbert Armond Perkins, Hampton Institute;
Mr. James Ellis Powell, Michigan Agricultural College;
Mr. Paul Klein Rees, Texas Technological College;
Mr. Warren Alonzo Rees, Texas Agricultural and Mechanical College;
Mr. Davis Payne Richardson, University of Chicago;
Mr. James Holmes Sturdivant, University of Texas;
Mr. Cecil Francis Whitaker, Macon, Ga.;
Professor Herman Zanstra, University of Washington.

Eleven applications for membership in the Society were received.

The Board of Trustees at their meeting revised the budget and transacted other financial business.

At the meeting of the Council it was announced that the committee on the Josiah Willard Gibbs Lecture had selected Dr. H. B. Williams, Professor of Physiology, College of Physicians and Surgeons, Columbia University to give the fourth Gibbs Lecture in Philadelphia at the time of the Annual Meeting and that his subject is *Mathematics and the Biological Sciences*. President Birkhoff appointed as committee on arrangements for this lecture, Professors F. H. Safford (chairman) and J. A. Miller and Miss Marion Reilly.

It was announced that President Birkhoff had appointed Vice-President Evans to represent the Society at the semi-centennial celebration of the Agricultural and Mechanical College of Texas on October 14–16, 1926.

Professor A. B. Coble was invited to give a Colloquium in connection with the Summer Meeting in Amherst in 1928.

A list of nominations for trustees, officers, and other members of the Council was adopted unanimously and ordered printed on the ballot.

President Birkhoff presided at the morning session, relieved in the afternoon by Professor O. D. Kellogg.

Titles and abstracts of the papers read at this meeting follow below. The papers of Douglas, Ettlinger, Franklin, Graves, Hollcroft, Moore, Rice, Shaub, Weisner, Whyburn, and Wilson were read by title. Professor Drach was introduced by Professor Fite, Professor LaMer by Dr. Gronwall, and Mr. Serghiesco by Professor Ritt.

1. Dr. B. O. Koopman: *On analytic solutions of differential equations in the neighborhood of non-analytic singular points.*

This paper will appear in an early issue of this Bulletin.

2. Professor J. F. Ritt: *A factorization theory for functions* $\sum_{i=1}^{n} a_i e^{\alpha_i x}$.

When, in the expression $a_0 e^{\alpha_0 x} + \cdots + a_n e^{\alpha_n x}$, we allow n to assume all positive integral values, and the a's and α's all constant values, we obtain a class of functions which is closed with respect to multiplication. The problem thus arises of determining all representations of a given function

of this class as a product of functions of this class. Simplicity of results is obtained, without loss of generality, by the assumption that $\alpha_0=0$, $a_0=1$. A function in which the α's all have rational ratios to one another is called simple. A function which is not the product of two functions is called *irreducible*. It is proved that every function can be expressed, in one and only one way, as a product of simple functions and of irreducible functions, the α's of every simple function having irrational ratios to those of every other.

3. Professor J. F. Ritt: *Simplification of Liouville's method in the theory of elementary functions.*

Liouville, about a century ago, studied the possibility of solving differential equations in finite terms. He was followed in this work by certain Scandinavian and Russian mathematicians. The present paper gives a much more powerful method than that of Liouville for handling questions of expressibility in finite terms. With the new method, the proofs of results already in the literature can be shortened from fifty to ninety percent. An illustration of the application of the method is contained in the Comptes Rendus for August 2, 1926.

4. Dr. B. F. Kimball: *Geodesics on a toroid.*

The surface studied is called a toroid, as it resembles a torus; it is, however, more general. The geodesics on this surface are described and classified. The conjugate points on each class of geodesics are also discussed, and some space is given to the description of the envelopes of the geodesics. The envelopes are investigated by calculating the second variation and then applying the theory of Lindeberg (Mathematische Annalen, vol. 59 (1904), p. 321). The method would seem to be a productive one.

5. Mr. S. Serghiesco: *Interpretation of the characteristic (in the Kronecker-Picard sense) of a particular system of four functions.*

Let $F_1(x, y, z)$, $F_2(x, y, z)$, $F_3(x, y, z)$, $F_4(x, y, z)$ be a system of four given continuous functions, $F_4(x, y, z)=0$ representing a certain closed surface S. It is known that the surface integral $I=(1/4\pi)\iint_S Adydz+Bdzdx+Cdxdy$, where A, B, C are special functions of $(F_1\cdots, F_{12}', \cdots, F_{1y}'\cdots, F_{12}'\cdots)$ is defined as the characteristic of F_1, F_2, F_3, and F_4. It is shown in this paper that when the four given functions are $F(x, y, z)$, $F_x'(x, y, z)$, $F_y'(x, y, z)$, $F_z'(x, y, x)$, the characteristic of these functions represents the total curvature, divided by 4π, of the whole closed surface S: $F(x, y, z)=0$. A geometrical verification is given through theorems of Gauss and definitions in differential geometry.

6. Dr. H. M. Gehman: *Concerning homogeneous plane continua.*

Mazurkiewicz has proved (Fundamenta Mathematicae, vol. 5 (1924), p. 137) that the only homogeneous plane continuous curves that exist

are the simple closed curve, the open curve, and the plane. It is not known whether there exist plane continua which are not continuous curves. The following theorems give some information concerning the nature of such continua if they do exist. (1) If a homogeneous plane continuum contains the interior of any circle, it contains all points of the plane. (2) If one point of a homogeneous plane continuum M is a cut-point, then M is an open curve. (3) A homogeneous plane continuum which is an irreducible continuum between two points is indecomposable.

7. Mr. W. L. Ayres: *A new characterization of plane continuous curves.*

This paper will appear in full in an early issue of this Bulletin.

8. Mr. V. B. Teach: *On the properties of fields for the Lagrange problem in parametric form.*

For the problem of Lagrange in the calculus of variations, when stated in the parametric form in the usual manner, a field analogous to that of Mayer for the problem in non-parametric form is defined. Theorems analogous to those on the structure of a field for the problem in non-parametric form are then derived, and the construction of fields and some properties of transversal surfaces discussed.

9. Dr. Jesse Douglas (National Research Fellow): *The transversality relative to a surface* $\int F(x, y, z, y', z')dx = \text{minimum}$.

This paper has appeared in the November-December number of this Bulletin.

10. Dr. Jesse Douglas (National Research Fellow): *A characteristic property of minimal surfaces.*

This paper has appeared in the November-December number of this Bulletin.

11. Professor R. L. Moore: *Concerning paths that do not disconnect a given continuous curve.*

It is shown that if, in a plane S, M is a bounded continuous curve, A and B are points of $S-M$ which do not lie in the same complementary domain of M, and K is the point set obtained by adding together all simple closed curves which lie in M and separate A from B, then (1) K is closed, and (2) there exists a simple continuous arc from A to B which does not disconnect M, that is to say, there exists an arc AB such that $M-(AB)\cdot M$ is connected.

12. Professor W. A. Wilson: *On the separation of a plane by two irreducible continua.*

This paper is a generalization of A. Rosenthal's results. It is shown that the conclusion of his principal theorem is valid if the two continua C_1 and

C_2 have in common two point sets α and β, where $\alpha \cdot \beta = 0$, if each of the continua C_1 and C_2 is irreducible between any point a of α and any point b of β, and both C_1 and C_2 are decomposable or one of them is indecomposable and the other is not the union of two indecomposable continua. It is also shown that the frontier of each secondary region (Nebengebiet) is a part of some oscillatory set of C_1 or C_2 or is a part of the union of the oscillatory sets of C_1 and C_2 about a or b; consequently, such a frontier is either a continuum of condensation of C_1+C_2, an indecomposable continuum, or the union of two indecomposable continua.

13. Professor H. J. Ettlinger: *Note on continuity in several variables.*

This paper appears in full in the present issue of this Bulletin.

14. Professor L. M. Graves (National Research Fellow): *On the existence of the absolute minimum in space problems of the calculus of variations.*

Tonelli in his recent *Fondamenti di Calcolo delle Variazioni* has developed new and powerful methods for treating the problem of the absolute minimum. The present paper applies these methods, with some necessary alterations, to the ordinary or non-parametric integrals $I_C = \int f(x, y, y')dx$ in space of $k+1$ dimensions. The classes of curves in which the minimum is sought are those called "complete" by Tonelli, and are so general as to include cases of fixed or variable end points, and problems of Lagrange with finite side conditions. A case of the latter would arise from application of Hamilton's principle to problems in dynamics with holonomic constraints.

15. Mr. W. M. Whyburn: *The Green's function for systems of differential equations whose coefficients contain a parameter.*

This paper treats a system of n differential equations together with n linear boundary conditions connecting the values of the functions at two points of the interval of definition. The coefficients are continuous functions of a parameter and summable functions of the independent variable, and finally are uniformly bounded. A theorem due to Bôcher (this Bulletin, vol. 20 (1914), p. 1) on the reduction of the order of compatibility of a differential system is carried over, and this extended theorem is used to extend the author's previous results (Annals of Mathematics, (2), vol. 26 (1924), pp. 125–130) to more general systems. A matrix of $(n)^2$ Green's functions (Bounitzky, Journal de Mathématiques, (6), vol. 5 (1909), pp. 65–125) is set up, and the properties of these functions are discussed.

16. Dr. Louis Weisner (National Research Fellow): *A theorem concerning direct products.*

This paper appears in full in the present issue of this Bulletin.

17. Mr. L. H. Rice: *Compounds of Cayley products of determinants of higher class.*

Continuing the general subject treated in previous papers (Journal of Mathematics and Physics, Massachusetts Institute of Technology, vol. 5, Nos. 1 and 4) which concerned both ordinary determinants and those of higher class, the writer shows that the mth compound of a Cayley product of two determinants of any classes and any signancies is the like Cayley product of the mth compounds of the factor determinants, and, as a corollary, that the same is true of the adjoint of a Cayley product. The paper will appear in volume 6 of the same Journal.

18. Professor T. R. Hollcroft: *Singularities of the Hessian.*

This paper appears in full in the present issue of this Bulletin.

19. Mr. H. C. Shaub: *Rational involutorial transformations in S_4 which leave invariant ∞^4 quadric varieties.*

By considering the (2, 1) correspondence between two spaces (x) and (x') of four dimensions for which a hyperplane in (x') has for image a quadric variety in (x), the author reduces the involutorial transformations which leave invariant ∞^2 quadratic varieties to four distinct types.

20. Professor Philip Franklin: *The classification of quadrics in euclidean three- and n-space, by means of covariants.*

Professor MacDuffee has recently given (American Mathematical Monthly, vol. 33, p. 243) the first complete classification of conics in the euclidean plane in terms of invariants and covariants. He uses the methods of the Lie theory. In this paper we give a complete classification of quadrics in n-space in terms of covariants. The system of covariants here used forms a Lie complete system, though our method of obtaining them is purely algebraic. The geometric significance of our results is also indicated.

21. Professor Philip Franklin: *The Simson lines of a triangle and the three-cusped hypocycloid.*

Steiner, in 1856, gave a discussion of the curve enveloped by the Simson lines of a triangle, and deduced many of its properties. He states (Werke, vol. 2, p. 642) that the curve is a three-cusped hypocycloid. While numerous analytical proofs of this have been given, the writer knows of no simple, direct synthetic proof in the literature. In this note such a proof is given. Certain related theorems are also obtained.

22. Professor G. D. Birkhoff: *On one-to-one transformations of surfaces.*

This paper deals with the structure of one-to-one analytic transformations T that do not possess an invariant area integral. The basis of the

treatment consists in the use of the author's extended form of the last geometric theorem of Poincaré, and of certain notions applicable to such transformations which are contained in a paper about to appear in the Göttinger Nachrichten.

23. Professor James Pierpont: *Classification of quadric surfaces in hyperbolic space.*

J. L. Coolidge (Transactions of this Society, vol. 4 (1903)) and T. J. Bromwich (the same Transactions, vol. 6 (1905)) have shown how to classify quadric surfaces in hyperbolic space. Coolidge's method rests on the discussion by Clebsch and Lindemann of the relative position of two quadric surfaces, while Bromwich's makes use of elementary divisors. Both methods are very neat, but require a considerable knowledge of geometry or algebra. A more elementary method has been employed by P. D. Schwartz to classify conics (see abstract in this Bulletin, vol. 32 (1926), p. 315). An extension of these principles to space leads to a classification of quadric surfaces.

24. Dr. C. M. Huber: *On complete systems of irrational invariants of associated point sets.*

The principal object of this paper is the determination of complete systems of invariants for sets of n points P_n^1 on a line. For $n=2p+2$, it is shown that all invariant products of the differences of the roots of the binary $(2p+2)$-ic are expressible in terms of elementary products linear in each root. The invariants for a set Q_{2p+2}^p defined by P_{2p+2}^1 are expressed in terms of the linear invariants for P_{2p+2}^1. For n odd it is conjectured that a complete system is made up of cyclic invariants. This is proved for n equal to 5 and 7, and a method is given for testing the validity of the conjectured theorem for any particular value of n. A mapping problem in connection with the cyclic invariants for P_5^1 and P_7^1 is discussed.

25. Professor Solomon Lefschetz: *Intersections and complete transformations of complexes and manifolds.* Second paper.

In this paper the author extends the results of his paper in the January, 1926, number of the Transactions of this Society to manifolds with a boundary when the topological behavior at all points of the latter is the same. It is found necessary to restrict somewhat the transformations considered. The results are surprisingly parallel to those for the no-boundary case.

26. Dr. M. H. Stone: *Expansions associated with a regular differential system of the second order.*

Associated with a differential system $u''+(\rho^2+g)u=0$, $0\leqq x\leqq 1$, $\alpha_i u^{(k_i)}(0)+\beta_i u^{(k_i)}(1)+\cdots=0$, $i=1, 2$, where g is a Lebesgue integral,

and the boundary conditions are in normal form and regular, is a formal series expansion for any function integrable Lebesgue or Denjoy. The series for the same function in connection with two such systems are said to be equivalent on an interval (a, b) when the difference of partial sums of the two series is $o(1)$ uniformly on (a, b). It is shown here that a necessary and sufficient condition for equivalence on (0, 1) of the series for an arbitrary summable function is that the boundary conditions of the two systems satisfy one of the following three relations: $k_1=\bar{k}_1=k_2=\bar{k}_2=1$; $k_1=\bar{k}_1=k_2=\bar{k}_2=0$; $k_1=\bar{k}_1=1$, $k_2=\bar{k}_2=0$, $\alpha_1\bar{\beta}_1-\bar{\alpha}_1\beta_1=0$, $\alpha_2\bar{\beta}_2-\bar{\alpha}_2\beta_2=0$. This condition is also shown necessary and sufficient for the equivalence on an interval (a, b) completely interior to (0, 1) of the two series formed for an arbitrary function integrable in the sense of Denjoy. Finally, the first order Cesàro mean of the series of any function integrable in the sense of Denjoy is examined.

27. Dr. T. H. Gronwall and Professor V. K. LaMer: *The variation of the dielectric constant in the Debye-Hückel theory.*

In this paper a method is developed for determining the ion sizes and the variation of the dielectric constant with the concentration, from measurements of vapor pressure and freezing-point lowering in a solution of a strong electrolyte.

28. Dr. T. H. Gronwall: *A diophantine equation connected with the hydrogen spectrum.*

It is shown that in the Rydberg-Ritz formula for the wave length of a line in the hydrogen spectrum, the same wave length may correspond to two different sets of quantum numbers.

29. Professor Jules Drach: *Determination of Liouville's elements of length with algebraic integrals for the equation of geodesics.*

If $ds^2=4\lambda(u, v)\,du\,dv$ is an element of length in symmetric coordinates, the determination of the geodesics requires integration of the partial differential equation $pq=\lambda$, where $p=\partial z/\partial u$, $q=\partial z/\partial v$. In Liouville's case, where $\lambda=F(\alpha)-G(\beta)$, with $\alpha=u-v$, $\beta=u+v$, we know the quadratic integral $\phi=p^2+q^2+2pq(F+G)/(F-G)$; another integral, giving the general geodesic line, is ψ defined by $d\psi=d\alpha/(\phi+4F)^{1/2}+d\beta/(\phi+4G)^{1/2}$ and expressed in ϕ, α, β. The author wishes to determine all forms of F and G for which another integral f, rational in p and q, exists. In these cases a certain function of ψ is an algebraic function of ψ and f. For simplicity it is supposed that f is a polynomial in p, q.

R. G. D. RICHARDSON,
Secretary.

ON THE METRIZATION PROBLEM AND RELATED PROBLEMS IN THE THEORY OF ABSTRACT SETS*

BY E. W. CHITTENDEN

1. *Topological Space.* In the theory of abstract sets we assume that we are given an arbitrary aggregate P and a relation between subsets of P which corresponds to the relation between a set and its derived set in the classical theory of sets of points.† That is, the mathematical concept abstract set in its current sense includes the notion limit point or point of accumulation. The introduction of limit points permits the definition of continuous 1-1 correspondence or homeomorphy. The study of such correspondences, particularly of invariants under homeomorphic transformations, constitutes the science of topology or analysis situs.‡ It seems proper therefore to speak of an abstract set as a topological space.§ Throughout this paper, the term *topological space* or *abstract set* refers to any system of the form (P, K) composed of an aggregate P and a relation of the form EKE' between the subsets E, E' of P which is subject to the condition, for every subset E of the aggregate P there is a unique set E' in the relation K to E. That is, the relation K defines a single-valued set-valued function on the class U of all subsets of the aggregate P.‖

* Presented to the Society by invitation of the program committee, at the Summer Meeting, Columbus, Ohio, September 8, 1926.

† See M. Fréchet, *Esquisse d'une théorie des ensembles abstraits*, Sir Asutosh Mookerjee's Commemoration volumes, II, p. 360, The Baptist Mission Press, Calcutta, 1922; *Sur les ensembles abstraits*, Annales de l'École Normale, vol. 38 (1921), p. 341ff.

‡ See H. Tietze, *Beiträge zur allgemeinen Topologie*, I., Mathematische Annalen, vol. 88 (1923), p. 290.

§ This terminology is suggested by Fréchet. See Comptes Rendus, vol. 180 (1925), p. 419.

‖ These functions are studied in detail in an unpublished article by the writer.

2. *The Metrization Problem.* The problem is to state in terms of the concepts point, and point of accumulation the conditions that a topological space be metric.

A metric space* is any topological space in which the points of accumulation are defined or definable in terms of a function (p, q) called the distance between the points p and q and satisfying the fqllowing conditions.

(0) *The distance (p, q) is a definite real number for every pair of points p, q.*

(1) *Two points are coincident if and only if their distance is zero.*

(2) *For any three points p, q, r,*

$$(p,q) \leqq (p,r) + (q,r).$$

It follows readily from these conditions that the distance (p, q) is non-negative, and that it is symmetric in p and q, $(p, q) = (q, p)$.† In a metric space a point p is a point of accumulation of a set E provided its distance from a variable point of E which is distinct from p has the lower bound zero.

The following illustrations convey some notion of the scope of the concept metric space. If the aggregate P denotes the linear continuum of all real numbers and $(p, q) = |p-q|$, the resulting space is metric. Similarly euclidean space is also metric. The Hilbert‡ space of infinitely many dimensions in which the coordinates $x_1, x_2, x_3, \cdots, x_n, \cdots$ of each point are subject to the condition that the sum of their squares be a convergent series is a metric space in which distance is defined by the formula

$$(p,q) = [(x_1-y_1)^2 + (x_2-y_2)^2 + \cdots + (x_n-y_n)^2 + \cdots]^{1/2},$$

* F. Hausdorff, *Grundzüge der Mengenlehre*, Leipzig, 1914, p. 211. The definition of distance is due to Fréchet, *Sur quelques points du calcul fonctionnel*, Rendiconti di Palermo, vol. 22 (1906).

† A. Lindenbaum, *L'espace métrique*, Fundamenta Mathematicae, vol. 8 (1926), pp. 209–222.

‡D. Hilbert, Göttinger Nachrichten, vol. 8 (1906). Hausdorff, loc. cit., p. 287.

in which the x_n, y_n are the coordinates of p and q, respectively.

If we take for our aggregate P the class of all functions $p=f(x)$, on an interval $a \leqq x \leqq b$, whose squares are summable, and define distance by the formula

$$(p,q) = \left[\int_a^b [f(x) - g(x)]^2 \, dx\right]^{1/2},$$

we obtain a metric space. It is necessary to make the convention that two points coincide if the corresponding functions differ only at a set of points of measure zero. The relation $p=Lp_n$ corresponds to convergence in the mean for the sequence of functions $p_n=f_n(x)$.

3. *Hausdorff Spaces.* A remarkable and important class of topological spaces has been defined by F. Hausdorff.* In a Hausdorff space the points of accumulation are defined in terms of a family of neighborhoods U conditioned by the following four postulates.

(A) *To every point p there corresponds at least one neighborhood U, and each neighborhood of p contains p.*

(B) *There is a neighborhood of a point p common to every two neighborhoods of p.*

(C) *If q is any element of a neighborhood U of a point p, then U contains all the points of a neighborhood of q.*

(D) *If p and q are distinct points, there exist neighborhoods of p and q, respectively, which have no common elements.*

The topological spaces of Hausdorff are evidently included among the classes (V) of Fréchet† in which the postulated family of neighborhoods is subject only to the condition (A). In a class (V) a point p is a point of accumulation of a set E in case every neighborhood of p contains a point of E distinct from p.

* Loc. cit., p. 212.

† *Esquisse*, p. 346; Bulletin des Sciences Mathématiques, (2), vol. 42 (1918), pp. 1–19.

It is easy to show that every metric space is a Hausdorff space. Let $S(p, a)$ denote the generalized sphere composed of all the points q of a metric space which satisfy the inequality $(p, q) < a$. The family of all such spheres of center p and radius a has the four properties of Hausdorff and evidently defines the same points of accumulation as the distance (p, q).

The following set of properties form a necessary and sufficient condition that a topological space be a Hausdorff space.*

(I) $(A+B)' = A' + B'$.

(II) *A set containing but a finite number of points has no point of accumulation.*

(III) *The derived set of every set is closed.*

(IV) *If p and q are any two distinct points there exist open sets U and V which are disjoined and contain p and q respectively.*

To complete the solution of the metrization problem we need only add the conditions that a Hausdorff space be metric.

4. *Existence of Non-Constant Continuous Functions.* The metrization problem is included in another problem proposed by Fréchet in correspondence with Paul Urysohn and with me. It is evident that the distance (p, q) of two points p and q is a continuous function of its arguments, and is not constant in a space of two or more points. Thus the metrization problem is related to the more general problem, under what conditions does a topological space admit the existence of a non-constant continuous function.† The topological conditions for the existence of such functions in a Hausdorff space have been dis-

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* *Esquisse*, p. 367.

† The definition of continuous function for general topological space is given by Fréchet in the following form: *A point p of space is interior to a set I if it belongs to I and is not a point of accumulation of any subset of the of the complement of I, $P-I$. A function $f=f(p)$ is continuous at a point p if the oscillation of the function f on the sets I to which p is interior has the lower bound zero.* **Esquisse**, p. 363.

covered by Urysohn.* I have recently succeeded in formulating these conditions for topological spaces in general.

The following form of the question regarding the existence of non-constant continuous functions is of particular importance in the present discussion. Characterize those spaces of Hausdorff in which it is possible to define for every two disjoined closed sets A and B a continuous function $f(p)$ which is equal to zero on A, one on B, and satisfies the inequality

$$0 \leqq f(p) \leqq 1$$

everywhere else, that is, on the set $C=P-A-B$.

We shall show that it is both necessary and sufficient for the space to be *normal*. A space is normal provided every pair of closed disjoined sets A and B is separated by open sets, that is, that there exist open sets U, V which are disjoined and include A and B respectively.

Since the condition is evidently necessary we proceed to the proof of its sufficiency. Let P be a normal Hausdorff space and let A and B be any two disjoined subsets of P. From the hypothesis of normality there exist two open sets U_1, V_1 such that

$$A = U_1, \qquad B = V_1, \qquad U_1V_1 = 0.$$

The set $P-U_1$ is closed and includes V_1. It follows that there exist disjoined open sets U_0, V_0 such that

$$A \leqq U_0, \qquad P - U_1 \leqq V_0.$$

Therefore†

$$A \leqq U_0 \leqq U_0^0 \leqq U_1, \qquad B \leqq P - U_1.$$

Since U_0^0 and $P-U_1$ are disjoined closed sets there is an open set $U_{1/2}$ for which

$$A \leqq U_0^0 \leqq U_{1/2} \leqq U_{1/2}^0 \leqq U_1.$$

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* *Über die Mächtigkeit der zusammenhängenden Mengen*, Mathematische Annalen, vol. 94 (1925), p. 290.

† $U^0 = U + U'$.

It follows readily that there exists for any positive integer n a series of the form

$$A \leqq U_0^0 \leqq U_{1/2^n} \leqq U_{1/2^n}^0 \leqq \cdots \leqq U_{m/2^n} \leqq U_{m/2^n}^0 \leqq \cdots \leqq U_1$$

where $B = P - U_1$.

If $x = Lr_n$, where $r_n = m_n/2^n$, and $0 \leqq r_n \leqq r_{n+1} \leqq x \leqq 1$, we set

$$U_x = \sum_{n=1}^{\infty} U_{r_n}.$$

By definition $U_x \leqq U_{x'}$ if $x < x'$. Furthermore $U_x^0 \leqq U_{x'}$. For if n and m are chosen so that $x < m/2^n < (m+1)/2^n < x'$, then

$$U_x^0 \leqq U_{m/2^n}^0 \leqq U_{(m+1)/2^n}^0 \leqq U_{x'}.$$

Let $L_0 = U_0$; $L_1 = P - U_1$; $L_x = U_x^0 - U_x$, $0 < x < 1$. Then the sets $L_x (0 \leqq x \leqq 1)$ are closed, and if $x \neq x'$, $L_x L_{x'} = 0$. The function $f(p)$ which is equal to x when p is a point of L_x has the required properties.

5. *Perfectly Separable Spaces.* Among the spaces which were considered by Hausdorff are those whose points of accumulation are definable in terms of an enumerable family of neighborhoods.* Such spaces are said to satisfy the second axiom of enumerability. Tychonoff and Vedenissof have called them separable spaces.† In a letter to me Fréchet calls attention to the fact that the word separable is already in use in a more general sense and suggests the term perfectly separable. The following important and remarkable theorem was discovered by Urysohn.‡

THEOREM. *A necessary and sufficient condition that a perfectly separable Hausdorff space be metric is that it be normal.*

It is easy to show that every metric space is normal.§ Let A and B be any two disjoined closed sets and let a_p denote the

* Loc. cit., p. 263.

† Bulletin des Sciences Mathématiques, vol. 50 (1926), p. 17.

‡ Mathematische Annalen, vol. 94 (1925), p. 309.

§ Tietze, loc. cit., p. 311.

lower bound of the distance (p, q) as q varies over B. Let U be the sum of the spheres $S(p, a_p/3)$ obtained by considering all the points of A. Let V denote the corresponding open set enclosing B. Then U and V are disjoined. For if r is a point common to U and V there exist points p and q such that $(p, r) < c/3$, $(q, r) < c/3$, where c is the greater of a_p, b_q. Then we have

$$(p, q) \leqq (p, r) + (r, q) < 2c/3,$$

contrary to the befininition of a_p b_q.

To show that the condition is sufficient let

$$U_1, \quad U_2, \quad U_3, \cdots, U_i, \cdots, U_j, \cdots$$

represent an enumerable set of neighborhoods determining the points of accumulation of a *normal* Hausdorff space P. For each of the enumerable family of pairs of neighborhoods U_i, U_j, such that $U_i^0 \leqq U_j$, there is a continuous function $f=f(p)$ which satisfies the conditions $f(p)=0$ on U_i^0, $f(p)=1$ on $P-U_j$, $0 \leqq f(p) \leqq 1$ on P. Let the set of all such functions be represented by the sequence

$$f_1, f_2, f_3, \cdots, f_n, \cdots.$$

Consider the function

$$(p, q) = \sum_{n=1}^{\infty} [f(p) - f(q)]/2^n.$$

This function is continuous because it is the sum of a uniformly convergent series of continuous functions. It is furthermore evident that $(p, p) = 0$, and that $(p, q) \leqq (p, r) + (q, r)$.

It is necessary to show that if $p \neq q$ then $(p, q) > 0$. Let U_j be a neighborhood of p which does not contain the point q. Since the sets $A = p$ and $B = P - U_j$ are closed and disjoined there is a neighborhood U_i such that $U_i^0 = U_j$. Let $f_n(p)$ be the continuous function corresponding to the two sets U_i, U_j. Since $f_n(p) = 0$ and $f_n(q) = 1$, we have $(p, q) \geqq 1/2^n$, which was to be proved.

It remains to show that the neighborhoods U_i and the distance (p, q) just defined determine the same space. This will

be the case if for each point p every neighborhood of p contains a sphere $S(p, a)$ and if every such sphere contains a neighborhood of p. Since the distance (p, q) is continuous in q it follows that a neighborhood of p, U_i, can be found on which the oscillation of the function (p, q) is less than $a/2$ for any value of a (>0). Hence the sphere $S(p, a)$ includes the neighborhood U_i.

Suppose that there is a point p and a neighborhood U_j of p such that for every positive value of a there is a point q_a of the sphere $S(p, a)$ which is exterior to U_j. As before there exists a neighborhood U_i of p such that $U_i \leqq U_j$. Let f_n be the function associated with the pair of neighborhoods U_i, U_j. Then $f_n(q) = 1$, and $(p, q) = 1/2^n$. This is impossible if $a < 1/2^n$.

6. *Axioms of Separation.* The property of normality is the third of a series of four axioms of separation which have been discussed in detail by Tietze.* Two point sets A and B are separated by open sets U, V if

$$A = U, \qquad B = V, \qquad UV = 0.$$

We consider the following four cases.

(1) *The sets A and B each contain one point only.*

(2) *The set A contains a single point, the set B is closed.*

(3) *A and B are any closed sets.*

(4) *A and B are disconnected. That is, neither set contains a point of accumulation of the other.*

The first of these axioms of separation is Axiom (D) of the set defining a Hausdorff space, and coincides with the fourth of the conditions that a topological space be a space of Hausdorff. We indicate the axiom which corresponds to case i by the symbol (IV_i). In the terminology of Paul Alexandroff and Urysohn† a space satisfying axiom (IV_2) is regular; (IV_3) is normal; (IV_4) is completely normal. Each of these axioms

* Loc cit., p. 300, etc.

† Mathematische Annalen, vol. 92 (1924), p. 263.

is stronger than its predecessor and independent of it. Metric spaces are completely normal. However there exist spaces which are completely normal but not metric.*

7. *Regular Spaces.* Is a regular perfectly separable Hausdorff space normal? This question was proposed by Urysohn† and answered in the affirmative by Tychonoff.‡ Let A and B be any disjoined closed sets and let

$$U_1, U_2, U_3, \cdots, U_m, \cdots$$

be an enumerable system of open sets defining the points of accumulation of a Hausdorff space P. For each point p of the set A there is a neighborhood U_{n_p} of p which with its derived set contains no point of B. The class of all the neighborhoods U_{n_p} determined by the points of A and the set B forms a sequence

$$V_1, V_2, V_3, \cdots, V_n, \cdots$$

of open sets whose sum includes the set A. In similar fashion we define a sequence of open sets

$$W_1, W_2, W_3, \cdots, W_n, \cdots$$

hose sum includes B, such that $W_n^0 A = 0$, $n = 1, 2, 3, \cdots$. Let $G_1 = U_1$, $H_1 = V_1 - G_1^0$, and in general,

$$G_n = U_n - \sum_{i=1}^{n-1} H_i^0, \qquad H_n = V_n - \sum_{i=1}^{n} G_i^0.$$

The sets G_n and H_n are evidently open. If we now set

$$G = \sum_{n=1}^{\infty} G_n, \qquad H = \sum_{n=1}^{\infty} H_n,$$

we can show that

$$A \leqq G, \qquad B \leqq H, \qquad GH = 0.$$

Since no point of A is contained in any set V_n, $G_n A = U_n A$, and therefore $A \leqq G$. Similarly $B \leqq H$. Suppose that there is a point p common to G and H. Then there must exist indices

* Tietze, loc. cit.

† Mathematische Annalen, vol. 94 (1925), p. 315.

‡ Mathematische Annalen, vol. 95 (1926), p. 139.

m and n for which p is common to the sets G_m, H_n. If $m \leqq n$, we have

$$G_m H_n = G_m\left(V_n - \sum_{i=1} G_i^0 \right) \leqq G_m(V_n - G_m^0) = 0,$$

a contradiction. A similar contradiction is obtained if $m > n$.

This result when combined with the theorem of the preceding section gives the fundamental theorem.

THEOREM. *A necessary and sufficient condition that a perfectly separable Hausdorff space be metric is that it be regular.*

8. *An Axiom of R. L. Moore.* The importance of the regular and perfectly separable, therefore metric, spaces in the analysis of continua is indicated by the fact that nine years before the publication of the discoveries of Urysohn, R. L. Moore assumed these properties in the first of a system of axioms for the foundations of plane analysis situs.* This axiom is furthermore of particular interest historically since it yields when slightly modified a necessary and sufficient condition that a topological space be metric and separable. The modified axiom of Moore may be stated as follows.

AXIOM (R. L. Moore). *We are given a space P in which point of accumulation is defined in terms of a family of classes of points called regions. Among the regions there exists a fundamental enumerable sequence*

$$R_1, R_2, R_3, \cdots, R_n, \cdots$$

with the following properties: (0) *for every region R there is an integer n such that R_n is a subset of R*; (1) *for every point p and integer n there is an integer n' greater than n such that $R_{n'}$ contains p*; (2) *if p and q are distinct points of a region R there is an integer m such that if n is greater than m and R_n contains p, then R_n is a subset of $R-q$.*†

* *On the foundations of plane analysis situs*, Transactions of this Society, vol. 17 (1916), pp. 131–164. The hypothesis of regularity was also made (apparently independently) by L. Vietoris, Monatshefte, vol. 31 (1921), p. 176.

† This differs from Axiom 1 (loc. cit.) with respect to the first sentence and condition (0) only.

It is easy to show that in a space satisfying this axiom the regions are open sets, and that the space is a regular and perfectly separable Hausdorff space, therefore metric and separable. The converse of this proposition is also true and is established by the following chain of propositions.

The axiom of Moore is a topological invariant. That is, if the axiom is satisfied in one of two homeomorphic spaces it is satisfied in the other. Furthermore if the axiom holds in a space P it holds in any relative subspace of P.

It is easy to show that every compact metric space admits the axiom. Since Urysohn* has shown that every separable metric space is homeomorphic with a subset of a compact domain in the Hilbert space it follows immediately that every separable metric space satisfies this axiom.†

THEOREM. *A necessary and sufficient condition that a topological space be metric and separable is that it satisfy the axiom of R. L. Moore.*‡

9. *Metrization of Compact Spaces.* It can now be shown that a compact and perfectly separable Hausdorff space is metrizable. It is sufficient to show that it is regular. We have to show that for each point p and closed set A there are open sets V, U which separate p and A. Enclose each point of a given closed set A in an open set which does not contain p and consider the system of open sets thus obtained together with the open set $P-A$. Since every compact perfectly separable space has the "any-to-finite"§ property of Borel, a finite subset of these open sets may be selected which covers the closed set A. Let

$$U_1, U_2, U_3, U_4, \cdots, U_n$$

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* Mathematische Annalen, vol. 92 (1924), p. 302.

† I have obtained a direct proof of this proposition.

‡ The fact that Axiom 1 is a sufficient condition for metrizability was inferred by R. L. Moore from the theorem of Tychonoff in §7 above.

§ The phrase was introduced by T. H. Hildebrandt in an article on *The Borel theorem and its generalizations*, this Bulletin, vol. 32 (1926), pp. 423–474.

be a finite family of open sets whose sum U includes A but not p. Then there is a neighborhood V of p which contains no point of the open set U. The proof of the following theorem of Urysohn* is now easily completed.

THEOREM. *A necessary and sufficient condition that an infinite compact Hausdorff space be metrizable is that it be perfectly separable.*

10. *The Metrization Conditions in Locally Compact Spaces.* The result just obtained has been applied by Alexandroff† to locally compact Hausdorff spaces. A space is locally compact provided there is for every point p a neighborhood V such that V^0 is compact. The following fundamental theorem may be stated.

THEOREM. *A necessary and sufficient condition that a locally compact Hausdorff space be metrizable is that the space be perfectly separable or else be the sum of a set (of arbitrary cardinal number) of disjoined domains which are perfectly separable subspaces of the given space.*

The proof that this condition is sufficient follows lines indicated previously. The essential part of the proof that the condition is necessary consists in showing that every locally compact metric space which is not perfectly separable is representable as a sum of disjoined perfectly separable spaces.

In a locally compact metric space every point p is the center of a sphere $S(p, a)$ which is compact and therefore perfectly separable. Let $G_0 = S(p_0, a_0)$ and assume that G_n is defined and perfectly separable. Then G_{n+1} is defined to be the sum of all the spheres $S(p, a)$ which contain a point of G_n. The set

$$G = G_0 + G_1 + \cdots + G_n + \cdots$$

is perfectly separable. If p_0 is replaced by any point of G the process just defined will lead to the same set G. Thus the

* Mathematische Annalen, vol. 92 (1924), pp. 275–293, and vol. 94 (1925), p. 313.

† Mathematische Annalen, vol. 92 (1924), pp. 294–301.

given space admits a decomposition into disjoined perfectly separable metric subspaces.

This theorem has the following interesting corollary.

COROLLARY. *A necessary and sufficient condition that a connected and locally compact space be metrizable is that it be perfectly separable.*

11. *Relations between Metrization and the Existence of Continuous Functions.* We have seen that the metrization problem is related to the more general problem of the existence of non-constant continuous functions and how the study of the latter problem led Urysohn to the solution of the metrization problem for perfectly separable spaces. It is therefore of interest to formulate the conditions that a space be metrizable in terms of continuous functions.

THEOREM. *A necessary and sufficient condition that a topological space which is equivalent to a class (V) of Fréchet be metrizable is that there exist a family of equally continuous functions with the following properties:*

(1) *each function of the family is defined and continuous throughout the space;*

(2) *for each point p there is a function of the family which vanishes at p and is bounded from zero on the complement of every neighborhood of p.*

To show that the condition is necessary, let P be a metric space and define the required family of functions to be the class of all functions $\phi(p) = (p, q)$, where (p, q) is the distance from p to q, the point q is held fixed and the point p allowed to vary. That this family of functions has the required properties is an immediate consequence of the conditions satisfied by the distance between two points.

It will now be shown that the condition is sufficient. The definition of distance is derived from the formula

$$(p,q) = \lim \sup \frac{|\phi(p) - \phi(q)|}{1 + |\phi(p) - \phi(q)|},$$

where all functions ϕ of the given equally continuous family are to be considered. It is evident that the function (p, q) just defined has the properties of distance. It remains to consider the relation between distance and point of accumulation.

Suppose that a point p is a point of accumulation of a set E. Because of the equal continuity of the functions ϕ there is for every positive number e a neighborhood V of p on which the oscillation of each function is less than e. Therefore the distance of the point p from the set E is zero.

Conversely, suppose that every sphere $S(p, a)$ of center p and radius a in the metric space just defined contains a point q of a set E. Let V be any neighborhood of the point p in the given space (V). By the second condition of the theorem there is a function ϕ and a number e such that $|\phi| > e$ on the set $P - V$. This implies that $(p, q) > e/2$ for all points q of $P - V$. Therefore the sphere $S(p, e/2)$ is contained in the neighborhood V. That is, V contains a point of E, and the point p is by definition a point of accumulation of E.

12. *The Equivalence of Distance and Uniformily Regular Écart.* Attempts have been made by Fréchet,* E. R. Hedrick,† A. D. Pitcher, and the writer‡ to obtain effective generalizations of the theory of metric spaces. That is, to impose hypotheses which yield substantially the same group of theorems about point sets and are less restrictive. It has however been established in each case that the conditions proposed imply that the resulting space is equivalent to a metric space.

In the theory proposed by Fréchet, distance is replaced by a function (p, q) with the properties (0), (1) of distance and the further property

(2′) *There is a function $f(e)$, approaching zero with e, such that for any three points p, q, r,*

$$(p,r) \leqq e, \qquad (q,r) \leqq e, \qquad \textit{imply} \quad (p,q) \leqq f(e).$$

* *Sur quelques points du calcul fonctionnel*, Rendiconti di Palermo, vol 30 (1906), p. 1–74.

† Transactions of this Society, vol. 12 (1911), pp. 285–294.

‡ Transactions of this Society, vol. 19 (1918), pp. 66–78.

The discovery by H. Hahn* that every space of this type which contains two or more distinct points admits continuous non-constant functions led Fréchet† to the conclusion that this class of spaces is metrizable. This conclusion was verified by Chittenden.‡

From this and other considerations Fréchet§ has in his later papers employed the term écart to refer to a non-negative single-valued real-valued function of two points. In this terminology the function (p, q) which was formerly called a voisinage becomes a uniformly regular écart.

THEOREM. *The concepts uniformly regular écart and distance are equivalent.*

Since distance is a uniformly regular écart for which $f(e) = 2e$ it is sufficient to show that a class of equally continuous functions satisfying the conditions of the theorem of §11 above may be defined in any space admitting a definition of its points of accumulation in terms of a uniformly regular écart.

If the given space P is singular the theorem is obvious. If it contains at least two points a number $a > 0$ can be chosen with the property: for every point p there is a point q such that $(p, q) > a$. This number a may be determined in the following manner. Let q', q'' be any two distinct points and let a be chosen so that $f(a) < (q', q'')$. Then one of the points q', q'' is effective as the required point q. For if $(p, q') \leqq a$, and $(p, q'') \leqq a$, then by the property $(2')$

$$(q', q'') \leqq f(a),$$

contrary to the definition of a.

Suppose $a' < a$ chosen so that $f(a') < a$. Let p_0 be any fixed point and let‖

$$A = S(p_0, a'), \quad B = P - S(p_0, a), \quad C = P - A - B.$$

* Monatshefte, vol. 19 (1908), pp. 247–257.

† Rendiconti di Palermo, vol. 30 (1910), p. 22–23.

‡ Transactions of this Society, vol. 18 (1917), pp. 161–166.

§ See Bulletin des Sciences Mathématiques, (2), vol. 42 (1918), pp. 1–19.

‖ The validity of the following discussion is unimpaired in case the set C or any of its subdivisions is a null set.

The sets A^0, B^0 are disjoined. Suppose the contrary. Then there would exist points p, q, r of A, B, C, respectively, such that $(p, r) < a''$, $(q, r) < a''$, where $f(a'') < a'$. Then since $(p_0, p) < a'$, and $(p, q) \leqq f(a'') < a'$, we have $(p_0, q) \leqq a$, a contradiction.

The set C will be divided into disjoined sets C_0, C_1 according to the following rule. If $(r, A) = (r, B)$, where (r, A) denotes the distance from the point r to the set A, the point r is assigned to C_0, otherwise to C_1. The sets A^0, C_1^0 are disjoined, likewise the sets C_0^0, B^0. It is sufficient to give the proof for the first case. Let a number c be chosen so that $f(c) < a''$, and assume that there is a point r common to A^0, C_1^0. Then points p and q of A and C_1 respectively exist such that $(p, r) < c$, $(q, r) < c$. Therefore $(p, q) \leqq f(c) < a''$. From the definition of C_1 there is a point q' of B such that $(q, q') < (p, q) < a''$. Therefore $(p, q') \leqq f(a'') < a'$ contradicting the definition of the set B.

From the sets A, B, C we obtain by iteration of this method of subdivision a development of the space P of which the mth stage has the form

$$A,\ C_{00\cdots 0},\ C_{00\cdots 01},\ C_{i_1 i_2 \cdots i_m},\ \cdots,\ C_{11\cdots 1},\ B,$$

in which the indices i assume the values 0, 1 only. This sequence has the further property that there exists a number a_m (which is independent of p_0) such that the distance of any two non adjacent sets exceeds a_m. It is quite easily shown by mathematical induction that the numbers a_m may be so chosen that

$$a > a_1 > a_2 \cdots,\ La_m = 0,$$

and that

$$a > f(a_1),\ a_1 > f(a_2),\ \cdots.$$

For each point p of C there is a unique set $C_{i_1 i_2 \cdots i_m}$ of stage m of which it is an element. Consequently each point determines a unique sequence of indices,

$$i_1,\ i_2,\ i_3,\ \cdots,\ i_m,\ \cdots,$$

and therefore corresponds to the number in the binary scale determined by these indices.

We proceed to the definition of a function ϕ of primary importance. On the set A, $\phi(p)=0$, on B, $\phi(p)=1$, on C

$$\phi(p) = \frac{i_1}{1} + \frac{i_2}{2^2} + \frac{i_3}{3^3} + \cdots .$$

This function is continuous. In fact, if $(p, q)<a_m$ then p and q lie in adjacent classes of the mth stage of the development of the space P and therefore

$$|\phi(p) - \phi(q)| \leqq 1/2^{m-1}.$$

Since the function ϕ is defined in terms of a and p_0, and the numbers a_m are independent of p_0, it follows that the family of all such functions obtained by varying p_0 and keeping a fixed is equally continuous.

If we now make the definitions

$$A_n = S(p_0, a_n), \qquad B_n = P - S(p_0, a_{n-1}),$$

$n=1, 2, 3, \cdots$, $a_0=a$, and let ϕ_n denote the function defined relative to A_n, B_n by the foregoing process, we obtain the desired family of equally continuous functions by considering all possible functions of the form

$$\phi = \sum_{n=1}^{\infty} \phi_n/2^n.$$

Since $\phi_n=1$ on B_n for each value of n, it follows that $\phi=1/2^n>0$ on the set $P-S(p_0, a_{n-1})$. Since $La_n=0$ the second condition of the theorem of § 11 is satisfied.

13. *Coherent Spaces.* Another attempt to generalize effectively the theory of metric spaces was made by A. D. Pitcher and E. W. Chittenden.* They considered an écart (p, q) in which the second condition on distance is replaced by the condition:

(2″) if $L(p,p_n)=0$, and $L(p_n,q_n)=0$, then $L(p,q_n)=0$.

A space in which the points of accumulation are definable in terms of a symmetric écart satisfying the condition (2″) is

* Loc. cit.

said to be coherent. It has recently been shown by Niemytski* that if the écart satisfies the further condition: $(p, q)=0$ is equivalent to $p=q$, then the space is metric.

14. *Spaces Defined by Developments.* The metrization problem for spaces whose points of accumulation are defined in terms of a development Δ was studied by Pitcher and Chittenden.† A development Δ is an arbitrary system of subclasses P^{ml} $(m=1, 2, 3, \cdots, l=1, 2, \cdots, l_m)$ of a fundamental class P in which the classes P^{ml} for a fixed index m form a stage Δ^m of the development.‡ The index l_m may have a finite or infinite range. If its range is finite for all values of m the development is said to be finite, otherwise infinite.

A point p is a point of accumulation of a set of points E relative to a development Δ provided there is a sequence of indices $m_1<m_2<m_3\cdots$ and a corresponding sequence of classes $P^{m_n l_n}$ each of which contains an element of the set $E-p$. The reader is referred to the original article for the details of this investigation. The metrization problem was solved for compact spaces. The paper contains a set of necessary and sufficient conditions that a compact Hausdorff space be metric.

15. *The General Metrization Problem.* The general problem, to determine the topological conditions for the metrization of a topological space, was first explicitly stated and solved by Alexandroff and Urysohn.§ It is however of interest to note that E. R. Hedrick‖ in continuing the search begun by Fréchet for a generalization of metric space discovered that a number of important theorems stated by Fréchet for classes

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* In an article to appear in the Transactions of this Society.

† Transactions of this Society, vol. 20 (1919), pp. 213–233.

‡ This definition is due to E. H. Moore, New Haven Mathematical Colloquium, Yale University Press, New Haven, 1910, pp. 1–150.

§ Comptes Rendus, vol. 177 (1923), p. 1274.

‖ Loc. cit.

(V) normales* could be proved in any class (L)† in which derived sets are closed, providing the given space admits a property called the *enclosable* property. While Fréchet‡ soon proved that the space thus defined by Hedrick was in fact a "classe (V) normale" (therefore metric), it is important to observe that a slight modification of the conditions imposed by Hedrick constitute a set of necessary and sufficient conditions for the metrization of an abstract set.

We shall give three solutions of the general metrization problem; the first contains a modified form of the enclosable property of Hedrick, the second is due to Alexandroff and Urysohn, and the third is based upon the notion of coherence introduced by Pitcher and Chittenden. These three sets of conditions are alike in requiring the existence of a type of development Δ of fundamental importance which it is proposed to call *regular*.

Each stage of a *regular* development of a topological space is a family Δ^m of open sets V^m which covers the space P. The development proceeds by consecutive stages, that is, each set V^{m+1} of the $(m+1)$st stage is a subset of a set of the mth stage. Furthermore, if

$$V^1, V^2, V^3, \cdots, V^m, \cdots$$

is any infinite sequence of open sets one from each stage of the development and if there is a point p which is common to the sets of the sequence, then that point is *determined* by the sequence. That is, if V is any neighborhood of the point p, then for some value of the integer m, we have $V^m = V$. The sets V^m of the mth stage of the development are said to be of *rank* m.

The following additional definitions will be needed. Two points p, q are *developed* of stage or rank m provided there is a set of rank m which contains them both. In a regular develop-

* A "classe (V) normale" is a class that is separable, perfect, admits a definition of voisinage and a generalization of the theorem of Cauchy.

† A class (L) is essentially a space in which point of accumulation is defined in terms of the relation limit of a sequence.

‡ Transactions of this Society, vol. 14 (1913), pp. 320–324.

ment two points which are developed of rank m are developed of any lower rank.

Two sequences p_m, q_m are connected by a regular development provided the points p_m, q_m are developed of stage $m(m=1, 2, 3, \cdots)$.

A regular development Δ will be said to be *coherent* provided the connection of sequences is transitive. That is, a sequence $\{p_m\}$ is connected with a sequence $\{q_m\}$ whenever there is a sequence $\{r_m\}$ such that $\{p_m\}$ is connected with V_m and $\{r_m\}$ is connected with $\{q_m\}$.

The $(m+1)$st stage of a development is said to be *inscribed* in the mth if every pair of sets of rank $m+1$ which have a common point is contained in a set of rank m.

THEOREM. *Let P be a Hausdorff space admitting a regular development Δ in open sets V^m. Each of the following three conditions is a necessary and sufficient condition that P be equivalent to a metric space.*

I. (Hedrick) *For any positive integer m there is an integer n such that for any point p there is a set V^m of rank m which includes all sets of rank n which contain p.*

II. (Alexandroff and Urysohn) *For each value of the integer m the $(m+1)$st stage of the development Δ is inscribed in the mth.*

III. *The development Δ is coherent.*

Since it is evident that the three conditions of the theorem are necessary we may proceed at once to the proofs of their sufficiency.

Let us consider first the condition of Hedrick. It will be convenient to assume that the class P is a set of zero rank and allow the index m to have the values $m=0, 1, 2, 3, \cdots$. Then from condition II there exists for each integer n an integer $m=g(n)$, the greatest value of m for which n is the integer determined by condition II. The function $g(n)$ is unbounded. Suppose the contrary. Then there would exist an integer m' such that for every value of n, $g(n)<m'$. But there is an inte-

ger n' determined by m', contrary to the definition of $g(n)$ as the greatest such integer.

The écart (p, q) of two points is defined as follows. We set $(p, p)=0$. If m is the largest integer for which p and q are developed of rank m then $(p, q)=1/2^m$. Evidently the écart is symmetric. And if $p \neq q$ then $(p, q)>0$. For if the points p and q are developed of every rank m, then there is a sequence of sets V^m, one from each stage of the development to which both are common, contrary to the definition of a regular development, and the fourth condition of Hausdorff.

It remains to be shown that the écart thus defined is uniformly regular and that it defines the space P.

Let n be any integer and let p, q, r be three points such that $(p, r)<1/2^n$, $(q, r)<1/2^n$. If $m=g(n)$ we have at once, from condition II, that p and q are developed of stage m and therefore $(p, q)<1/2^m$. If $1/2^{n-1}<e\leqq 1/2^n$, and we set $f(e)=1/2^m$ we obtain a function $f(e)$ satisfying the required condition (2').

It remains to show that the given space and the derived space are equivalent. It is evident that if every set V^m which contains p contains a point of a set E then the écart (p, E) is null. Conversely, if $(p, E)=0$ there is for every integer n a point q_n of E such that $(p, q_n)<1/2^n$. That is, p and q_n are common to some neighborhood V^n. It follows that q_n is an element of $V^{q(m)}(p)$. Since $Lg(n)=\infty$, it follows from the regularity of the development Δ that every neighborhood of p contains a point of E.

It will now be shown that condition II implies condition I. We assume a regular development Δ satisfying condition II. A regular development Δ_1 satisfying condition I for $n=m+1$ may be defined in terms of Δ as follows. Stage $\Delta_1{}^m$ of Δ_1 is the family of all open sets $U^m(p)$ obtained by forming the sum for each point p of all the sets V^m of rank m of Δ which contain p. To show that if a point p is common to a sequence of sets

$$U^1, U^2, U^3 \cdots, U^m, \cdots$$

then p is determined by that sequence, let V be any neighborhood of p, and suppose that for each value of m there is a point

q_m of U^m which does not belong to V. Since q_m is in the set U^m there must be a set V^m of rank m of which contains both p and q_m. But the sequence of sets V^m determines p. Therefore for some value of m, V^m (and therefore q_m) is contained in V, contrary to the hypothesis on q_m.

In the proof of the sufficiency of condition I the écart (p, q) of two points was defined, and without reference to that condition. This definition will be applied in the proof of the sufficiency of condition III. Because of the result obtained by Niemytski stated in § 13, we have only to show that this écart satisfies the condition (2''), since the equivalence of the spaces has already been established.

It is at once evident from the definition of écart that if two sequences are connected by the development then $L(p_n, q_n) = 0$ and conversely. If then, $L(p, p_n) = 0$, and $L(p_n, q_n) = 0$, the sequence composed of the repeated element p must because of condition III be connected with the sequence q_n. Therefore $L(p, q) = 0$, as required by condition (2'').

16. *The Borel Theorem.* A set E will be said to possess the "any-to-finite" form of the property of Borel in case every infinite proper covering F of E contains a finite proper covering. A family F of sets G is a proper covering of a set E if every element of E is interior to some set E. In a metric space a necessary and sufficient condition that a set E admit this property is that E be compact and closed. Thus the problem, to characterize those spaces in which the "any-to-finite" form of the property of Borel is equivalent to compact (or self-compact) and closed, is very closely related to the metrization problem. As Hildebrandt* has presented an excellent and full account of recent work on this problem, it is unnecessary to discuss it further here.

THE UNIVERSITY OF IOWA

* Loc. cit.

A THEOREM ON FACTORIZATION*

BY D. N. LEHMER

In a note in this Bulletin† I observed that if $R=pq$ is the product of two odd factors whose difference is less than twice the fourth root of R then the factors of R are obtainable directly from the expansion of $R^{1/2}$ in a continued fraction. This theorem comes from the fact that in view of a theorem due to Lagrange, $(p-q)^2/4$ will appear as a denominator of a complete quotient in that expansion, and that therefore the diophantine equation $x^2-Ry^2=(p-q)^2/4$ will have the integral solution $x=\frac{1}{2}(r+q)$, $y=1$.

The object of the present note is to point out that the method is of much wider application than the above statement would indicate. For consider the identity

$$\left(\frac{mp+nq}{2}\right)^2-\left(\frac{mp-nq}{2}\right)^2=mnpq.$$

From this it appears that if mn is a square and if m and n are both odd or both even, we will have an integral solution of the equation

$$x^2-Ry^2=\frac{1}{4}(mp-nq)^2,$$

namely

$$x=\frac{1}{2}(mp+nq), \qquad y=(mn)^{1/2}.$$

By Lagrange's theorem, therefore, if $mp-nq<2R^{1/4}$ one of the denominators in the expansion of $R^{1/2}$ will certainly be $(mp-nq)^2/4$ and since the numerator of the preceding convergent will be $(mp+nq)/2$ these two numbers will serve to furnish the factors p and q of R. We have then the following theorem.

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* Presented to the Society, San Francisco Section, October 30, 1926.

† Vol. 13 (1906-7), p. 501. Translated in Sphinx-Oedipe, 1911. Given also in Kraitchik's *Recherches sur la Théorie des Nombres*, p. 73.

THEOREM. *If $R=pq$ is the product of two odd factors, and if two numbers m and n, both even or both odd, are obtainable such that their product is a square and also such that $mp-nq<2R^{1/4}$ then the continued fraction for $R^{1/2}$ will furnish without trial the factors p and q of R.*

It should be noted that if the difference $mp-nq$ is less than the fourth root of R the restriction that m and n be both even or both odd may be disregarded, for in that case $2m$ and $2n$ are suitable multipliers. Also it is worth noting that the square denominator will appear in the complete quotient when the denominator of the preceding convergent is $(mn)^{1/2}$. This means that in the original theorem the desired square is under the third complete quotient.

An example will indicate the method of attacking a number by this method. Let $A=1564,08789$. The square root expansion gives the following series of denominators for the complete quotients: 1, 8753, 15013, 3740, 529, $\cdots$, the partial quotients being 12506, 2, 1, 6, 47, $\cdots$. The convergent preceding the complete quotient with the square denominator 529 is found to be 250127/20. We have then

$$(mp+nq)/2 = 250127, \qquad (mp-nq)/2 = 23.$$

Whence

$$mp = 250150, \qquad nq = 250104.$$

Since, now, $pq=A$ and $mn=20^2=400$ it is easily found that $p=31263$, $q=5003$, $m=8$, $n=50$. The success of the method was due to the fact that the difference $mp-nq=46$, which is less that $2A^{1/4}=222$. In this example also we see that $36p-225q=207$, which is also less than 222; but since here the values of m and n differ in parity, these values will not appear in the expansion. Similarly the difference $mp-nq=23$ for $m=4$, $n=25$, and m and n being different in parity these values will also not appear in the expansion. But since 23 is less than $A^{1/4}$, we will have $2m$ and $2n$ for suitable values, and these are indeed the ones that do appear.

THE UNIVERSITY OF CALIFORNIA

ON CONTINUITY IN SEVERAL VARIABLES*

BY H. J. ETTLINGER

The following theorem on the continuity of a function of several variables is contained implicitly in a theorem on the existence of solutions of differential equations by Carathéodory.† It is of a general nature and independent of the context in which it is found. It is, therefore, worth while isolating and signalizing it.

THEOREM. *Hypotheses*: (1) $f(x, [y])$ *is defined in the domain* $R: a<x<b$, $y_0-k<[y]<y_0+k$, *where* $[y]\equiv(y_1, \cdots, y_n)$.

(2) *For each* $[y]$ *in* R, $f_x'(x, [y])$ *exists almost everywhere on* $a<x<b$, *and is summable on* $a<x<b$.

(3) $|f_x'(x, [y])|\leqq M(x)$, *where* $M(x)$ *is summable on* $a<x<b$.

(4) *For every* x *on* $a<x<b$, $f(x, [y])$ *is continuous in* $[y]$ *at* $y_1=y_0$.

Conclusion: $f(x, [y])$ *is continuous in* $(x, [y])$ *at* $(x, [y_0])$ *where* x *is on* $a<x<b$.

PROOF. Let us put $F_1\equiv f(x+h,[y_0+k_i])$, where $|k_i|\leqq k$; $F_2\equiv f(x,[y_0+k_i])$; $F_3\equiv f(x,[y_0])$. From hypothesis (2), we have

$$F_1 - F_2 = \int_x^{x+h} f_t'(t, [y_0 + k_i])dt,$$

which becomes by hypothesis (3),

$$|F_1 - F_2| \leqq \int_x^{x+h} M(t)dt. \tag{1}$$

Also, from hypothesis (4), it follows that given a positive ϵ, there exists a positive number k'_{xy_0}, such that

$$|F_2 - F_3| < \epsilon, \quad \text{for} \quad |k_i| < k'_{xy_0}. \tag{2}$$

* Presented to the Society, October 30, 1926.

† *Vorlesungen ueber Reelle Funktionen*, Leipzig, 1918, p. 678, Satz 5.

On the other hand, since $F_1 - F_3 \equiv F_1 - F_2 + F_2 - F_3$. it follows that

$$|F_1 - F_3| \leqq |F_1 - F_2| + |F_2 - F_3|.$$

By applying (1) and (2) to this inequality, we obtain the result $f(x, [y])$ is continuous in $(x, [y])$ at $(x, [y_0])$, where x is any value on $a<x<b$.

Let $F(x, y)$ be defined in the interior of the square S: $a<x, y<b$, and have the following properties, (1) for a fixed y, it is summable in x on $a<x<b$, (2) $|F(x, y)| \leqq K(x)$, for all values of y on $a<y<b$, where $K(x)$ is summable on $a<x<b$, (3) for every fixed x except a null set, $F(x, y)$ is continuous in y. Consider

$$f(x, y) = \int_a^x F(t, y)dt.$$

From a well known theorem concerning an indefinite integral we have that $f(x, y)$ is continuous in x for y fixed. From a theorem first stated by Carathéodory,* $f(x, y)$ is continuous in y for fixed x. *From the theorem stated above,* $f(x, y)$ is continuous in (x, y) for all points in R.

The conclusion of the preceding paragraph is of importance in showing that if a system of first order differential equations which satisfy the conditions of Carathéodory's† existence theorem be solved by the method of successive approximations, not only will the solutions be continuous functions of the independent variable and the parameter when both vary arbitrarily (as given by Carathéodory's theorem) but also the approximating functions will have continuity of this same kind.‡

THE UNIVERSITY OF TEXAS

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* Loc. cit., p. 639.

† Loc. cit., p. 678.

‡ For a pair of linear first order differential equations, see Sturdivant, *Properties of second order linear systems with Lebesgue integrable coefficients*, offered to the Transactions of this Society.

ON A PROBLEM IN CLOSURE*

BY VIRGIL SNYDER

The following note concerns finite groups of birational transformations which leave an algebraic curve of genus 1 invariant.

Let the curve be expressed as a C_4 in S_3, intersection of two general quadric surfaces. This curve is invariant under the linear group G_8, of order eight, generated by the harmonic homologies defined by the self-conjugate tetrahedron associated with C_4. The points of C_4 are thus arranged in sets of 8, forming a linear I_8^1 of genus 0. If the curve be projected upon a plane from an arbitrary point, a plane quartic C_4 results with nodes at K_1, K_2. The four central homologies become four perspective quadratic involutions T_i with centers O_i not on C_4; the three axial involutions become three non-perspective quadratic inversions, with fundamental points O_{ik} for $T_iT_k = T_kT_i = T_{ki} = T_{im}$ at the diagonal points of the quadrangle $O_1O_2O_3O_4$. The nodes K_1, K_2 are the other fundamental points for all seven operations.†

From the theorem of Bertini it follows that in any plane nodal quartic one and only one conic can be found meeting

* Presented to the Society, September 8, 1926.

† A brief synthetic outline of the properties of G_8 was first given by C. Segre, *Su una trasformazione irrazionale dello spazio . . .* , Giornale di Matematiche, vol. 21 (1883), pp. 355–378. This was amplified in connection with a larger problem by D. Montesano, *Su alcuni gruppi chiusi di trasformazioni involutorie nel piano e nello spazio*, Atti Istituto Veneto, (6), vol. 6 (1888), pp. 1425–1444. It is also contained in the papers by K. Meister, *Ueber die Systeme, welche durch Kegelschnitte mit einem gemeinsamen Polardreieck, bez. durch Flächen zweiten Grades mit einem gemeinsamen Polartetraeder gebildet werden*, Zeitschrift der Mathematik und Physik, vol. 31 (1886), pp. 321–347; vol. 34 (1889), pp. 6–24; 73–91 and by H. E. Timerding, *Ueber die quadratische Transformation durch welche die Ebenen des Raumes in ein System von Flächen zweiter Ordnung mit gemeinsamen Poltetraeder übergeführt werden*, Annali di Matematica, (3), vol. 1 (1898), pp. 95–117.

any line through the node in two points which form with the residual intersections of the line and the quartic a harmonic set. Using this conic as conic of invariant points and the node as center in a perspective quadratic inversion, the quartic is transformed into itself. In case there are two nodes there is a conic of Bertini associated with each, and the conic associated with one node passes through the other. Let these operations be denoted by S_1, S_2. In a recent memoir* the question is raised whether the group generated by S_1, S_2 and G_8 can be finite, but the author does not answer it. The answer may be difficult if the method is restricted to pure geometry. By assistance of the parametric representation in terms of elliptic functions it can be answered in every case.

Let

$$F = x_1^2 - x_2^2 + x_3^2 - x_4^2 = 0, \qquad \sum a_i x_i^2 = 0$$

define the curve. The tetrahedron of reference is the self-conjugate tetrahedron of the pencil of quadrics, and the equations of the harmonic homology H_1 with center at vertex (1, 0, 0, 0), invariant plane $x_1=0$, are $\rho x_1' = -x_1$, $\rho x_i' = x_i$, $i=2, 3, 4$; similarly for H_2, H_3, H_4.

By projecting $F=0$ upon $x_4=0$ from (1, 1, 1, 1), the operation H_1 becomes the perspective inversion

$$(T_1) \qquad \begin{cases} x_1' = (x_2 - x_3)(x_1 - x_2 - x_3), \\ x_2' = x_2(x_1 - x_2 + x_3), \\ x_3' = x_3(x_1 + x_2 - x_3). \end{cases}$$

The center is $O_1 = (1, 0, 0)$, and the other fundamental points are

$$K_1 = (0,1,1), \qquad K_2 = (1,1,0).$$

The conic of invariant points is

$$(x_1 - x_2)^2 + 2x_1x_3 - x_3^2 = 0.$$

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* E. Ciani, *Le quartiche piane invertibili*, Giornale di Matematiche, vol. 57 (1919), 47 pages.

Similar expressions exist for T_i, $i=2, 3, 4$, and for T_{ik}, $O_4=(1, 1, 1)$. The triangle formed by any three of the points $O_i\ O_k\ O_l$ is self-conjugate as to the conic of invariant points associated with the fourth.*

Among the space quartics belonging to the same self-conjugate tetrahedron are ∞^1 through the center of projection. This pencil is projected into a pencil of plane cubics through K_1, K_2, the four vertices O_1, O_2, O_3, O_4, and the three diagonal points O_{ik}. Hence by suppressing any side of the quadrangle, the four remaining basis points and K_1, K_2 lie on a conic.

The equation can be reduced to the form

$$k_1x_1(x_2-x_3)(x_2+x_3-x_1)+k_2x_2(x_3-x_1)(x_1-x_2+x_3)=0.$$

The points K_1, K_2 form a pair of conjugate points in the Geiser net determined by the other seven basis points. The line joining them meets any C_3 of the pencil in just one point, which with the seven basis points makes a group of I_8 on that curve. Any position on C_3 can be chosen for one of them, and then the other is uniquely fixed.

The points O_i are all cotangential, and any one of them can be assumed at will. The fundamental points O_{ik} are collinear with O_iO_k and with O_lO_m. Associated with any point P on C_3 are the points of tangency of the four tangents from P. The other three from the first tangential of P have their points of contact at O_{ik}. Expressed in parametric form, assuming the form to which every elliptic cubic can be reduced, that three points are collinear when the sum of their parameters is congruent to zero, then if P has the parameter u, we have

$$O_1=-\frac{u}{2},\qquad O_2=-\frac{u}{2}+\omega,\qquad O_3=-\frac{u}{2}+\omega',$$

$$O_4=-\frac{u}{2}+\omega+\omega',\qquad O_{12}=u+\omega,\qquad O_{13}=u+\omega',$$

$$O_{14}=u+\omega+\omega'.$$

* The equations of the operations of the group and those of an ∞^2 family of binodal quartics each invariant under it were determined in my seminar by Miss Bertha I. Hart, of Western Maryland College.

Let K_1 have the parameter v; then K_2 has the parameter $-u-v$. A conic through K_1 and through the four points remaining of the seven basis points when three on any line have been suppressed will also pass through K_2.

The involution in which K_1, K_2 are a pair of conjugate points is that having the polar conic at P for conic of invariant points, and P for vertex.

If K_1, K_2 be regarded as fixed, then a cubic of the pencil is fixed by its residual intersection with K_1, K_2. Since P_1 K_1 can be chosen at will and satisfy all the conditions of the problem it follows that the group generated by K_1, O_1 is generally of infinite order. If finite, K_1, G_8 is also finite.

Since P can be chosen at will on C_3 and K_1 can be chosen at will, there are ∞^2 groups of I_8 in the plane. As groups of operations there appear in sets of pencils, those of any set (O_i on any curve of the pencil) having the three operations T_{ik} in common, so far as the transformations of points on the curve are concerned.

These results can be interpreted in terms of C_4 in space directly. By an appropriate linear transformation we may write*

$$x_1=\sigma_1(2u),\quad x_2=\sigma_2(2u),\quad x_3=\sigma_3(2u),\quad x_4=\sigma(2u),$$

the σ_i being those defined in the Schwarz-Weierstrass *Formeln und Lehrsätze*, pp. 21–2.

Four points with parameters $u_1, \cdots, u_4$ are collinear if $\Sigma u_i=0$. The operations of G_8 are defined as follows:

$$\begin{aligned} u' &= -u+\omega+\omega' && \text{is } H_1,\\ u' &= -u+\omega' && \text{is } H_2,\\ u' &= -u+\omega && \text{is } H_3,\\ u' &= -u && \text{is } H_4, \end{aligned}$$

from which the H_{ik} at once follow.

A quadric of the pencil through C passes through any given point P. The quadric contains a generator of each system,

* E. Lange, *Die sechzehn Wendeberührungspunkte der Raumcurve vierter Ordnung, erster Species*, Schlömilchs Zeitschrift, vol. 28 (1883), pp. 1–23.

through P, each meeting C_4 twice. If this quadric be projected stereographically from P, the plane quadric will have a double point at K_1 on one generator and at K_2 on the other. Consider the line PK_1. It contains P_1, P_2 on C_4 and any plane through PK_1 meets C_4 in two points P_3, P_4 which project into two points collinear with K_1. These points are interchanged by S_1, hence in space we may choose u_1 and u_2 at will, then pass a plane through the points determined by them and any third point u_3. The space operation consists in interchanging the point u_3 with the fourth point u_4 as the plane turns about u_1 u_2, as u_3 describes C_4.

The operation has the form

$$u' = -u + c,$$

where c is any given constant. If this be associated with H_i the product is not periodic unless c is of the form $r\omega + r'\omega'$ wherein r, r' are both rational. But if this product is rational, then the group generated by S and G_8 is finite. The operation S_2 is defined by

$$u' = -u - c,$$

hence the necessary and sufficient condition that S_1, S_2, G_8 generate a finite group is that S_1, S_2 generate a finite group.

CORNELL UNIVERSITY

A THEOREM CONCERNING DIRECT PRODUCTS*

BY LOUIS WEISNER†

The theorem in question may be stated as follows. *A group of order mn, where m and n are relatively prime, in which every element whose order divides m is commutative with every element whose order divides n, is the direct product of two groups of orders m and n.*

Burnside‡ has proved the theorem for the special case in which either m or n is a power of a prime. Hence, to prove the theorem, we need only show that it is true for groups of order mn if it is true for groups of order $<mn$. To avoid trivial cases we assume that m and $n>1$. We denote by m_1, m_2, and n_1, n_2, divisors >1 of m and n respectively.

If the group G contains an element of order m, let p be a prime factor of m and p^α the highest power of p that divides m. Every element of G whose order divides p^α is commutative with every element whose order is prime to p. It follows from the special case referred to, that G contains an invariant subgroup of order mn/p^α, which contains an invariant subgroup of order n, as every element whose order divides m/p^α is commutative with every element whose order divides n.

We suppose now that G contains no element of order m. The normaliser H of an element s of order m_1 includes all the Sylow subgroups of G whose orders divide n, and is therefore of order $m_2 n$, where $m_2 \geqq m_1$. If $m_2 < m$, H contains an invariant subgroup of order n. If $m_2 = m$, s is invariant under G. An element t of G corresponding to an element t' of $G/(s)$ of order n_1 is of order $n_1\mu$, where μ divides m_1. Hence G contains§ two elements t_1 and s_1 of orders n_1 and μ respectively, such that

* Presented to the Society, October 30, 1926.

† National Research Fellow.

‡ W. Burnside, *Theory of Groups*, 2d edition, p. 327.

§ Loc. cit., p. 16.

$t=t_1s_1$. Since $(t_1s_1)^{n_1}=s_1^{n_1}$ is in (s), and n_1 is prime to the order of s_1, s_1 is a power of s. Thus t_1, as well as t, corresponds to t' in the isomorphism of G with $G/(s)$; but t_1 and t' are of the same order n_1. Every element of $G/(s)$ whose order is a divisor of m/m_1 corresponds to an element of G whose order is a divisor of m. It follows that t', and hence every element of $G/(s)$ whose order divides n, is commutative with every element whose order divides m/m_1. Hence $G/(s)$, being of order $<mn$, contains an invariant subgroup of order n. The corresponding subgroup of G, being of order $m_1n<mn$, also contains an invariant subgroup of order n.

Thus in all cases G contains a subgroup of order n. Similarly G contains a subgroup of order m. G is evidently the direct product of these two subgroups.

HARVARD UNIVERSITY

A CUBIC CURVE CONNECTED WITH TWO TRIANGLES

BY H. BATEMAN

1. *Introduction.* If ABC, XYZ are two triangles, a cubic curve Γ_3 may be associated with them as follows.* Let (PQ, RS) denote the point of intersection of the lines PQ, RS; then Γ_3 is the locus of a point O such that (OA, YZ), (OB, ZX), (OC, XY) are collinear and also the locus of a point O for which (OX, BC), (OY, CA), (OZ, AB) are collinear. In fact when one set of three points is collinear the other set of three is also collinear. Take ABC as triangle of reference and let the points X, Y, Z have coordinates (x_1, x_2, x_3), (y_1, y_2, y_3), (z_1, z_2, z_3) respectively, then if (α, β, γ) are current coordinates

* H. Grassmann, *Die lineale Ausdehnungslehre*, 1844, p. 226. The corresponding quartic surface connected with two tetrahedra is mentioned by H. Fritz, Pr. Ludw. Gymn. Darmstadt [reference taken from Jahrbuch der Fortschritte der Mathematik, vol. 21 (1889), p. 725] and by C. M. Jessop, *Quartic Surfaces*, Cambridge, 1916, p. 189.

the condition for the collinearity of the second sets of three points may be expressed in the form

$$(\gamma x_1 - \alpha x_2)(\alpha y_2 - \beta y_1)(\beta z_3 - \gamma z_2) = (\alpha x_2 - \beta x_1)(\beta y_3 - \gamma y_2)(\gamma z_1 - \alpha z_3).$$

This equation represents a cubic curve passing through the points* A, B, C, X, Y, Z, (BC, YZ), (CA, ZX), (AB, XY), (CY, BZ), (AZ, CX), (BX, AY). There is evidently only one cubic curve passing through all these points, and since a cubic through all these points is obtained for the locus when the three points (OA, YZ), (OB, ZX), (OC, XY) are collinear, Γ_3 must be the locus in both cases.

This is easily verified analytically. If X_1, X_2, etc. denote the co-factors of the constituents x_1, x_2, etc. in the determinant

$$\Delta = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix},$$

the equation of the locus in the first case may be expressed in the form

$$\begin{vmatrix} \beta X_2 + \gamma X_3 & -\beta X_1 & -\gamma X_1 \\ -\alpha Y_2 & \alpha Y_1 + \gamma Y_3 & -\gamma Y_2 \\ -\alpha Z_3 & -\beta Z_3 & \alpha Z_1 + \beta Z_2 \end{vmatrix} = 0,$$

and the equivalence of the two equations is easily verified with the aid of a number of identities, one of which is

$$2X_1Y_2Z_3 - Y_1X_3Z_2 - Z_1X_2Y_3 = \Delta(2x_1y_2z_3 - y_1x_3z_2 - z_1x_2y_3) = \Delta(x_1X_1 + y_2Y_2 + z_3Z_3 - \Delta).$$

2. *Salmon's Projective Invariant.* By a well known theorem due to Salmon the cross ratio of the four tangents to a cubic from the point on the curve is constant and represents

* The properties of this configuration of 12 points and some degenerate cases of the cubic curve are discussed by H. M. Taylor, Proceedings of the London Society, (1), vol. 28 (1897), pp. 545–555.

an invariant of the curve. In the present case this invariant will be an invariant of the two triangles* and it seems worth while to calculate it. A direct calculation is laborious when use is made of the equation of the curve in one of the forms already given; the work may be simplified, however, by noticing that the four lines BC, YZ, BZ, CY form a complete quadrilateral whose six vertices lie on the curve. Denoting the equations of these lines by $\alpha=0$, $\theta=0$, $\phi=0$, $\psi=0$ respectively, the equation of the curve may be written in the form

$$\frac{A}{\alpha}+\frac{B}{\theta}+\frac{C}{\phi}+\frac{D}{\psi}=0,$$

where the identical relation between $\alpha, \theta, \phi, \psi$ is

$$\alpha+\theta+\phi+\psi=0,$$

and

$$A=y_2z_1^2X_2^2Z_3, \qquad B=y_1^2z_3X_2^2Y_2,$$
$$C=y_1^2z_1^2X_1\Delta, \qquad D=x_1X_2^2X_3^2.$$

If λ is one of the cross ratios of the four tangents, λ is connected with the coefficients A, B, C, D by the relation†

$$(\lambda+1)^2(\lambda-2)^2(\lambda-\tfrac{1}{2})^2I^3=27(\lambda^2-\lambda+1)^3J^2,$$

where I and J, the usual invariants of a biquadratic equation, are given by the equations

$$12I=\Theta^2-48ABCD, \quad 216J=\Theta[72ABCD-\Theta^2],$$
$$I^3-27J^2=A^2B^2C^2D^2[\Theta^2-64ABCD],$$

where

$$\Theta^2=A^2+B^2+C^2+D^2-2BC-2CA-2AB$$
$$-2AD-2BD-2CD.$$

* A complete set of invariants of two triangles is given in a paper by D. D. Leib, Transactions of this Society, vol. 10 (1909), p. 361. In our case there is a correspondence between the vertices of the triangles and the invariant theory is a little different.

† The coefficient m in the canonical equation $x^3+y^3+z^3+6mxyz=0$ is connected with I and J by the relation

$$\frac{64m^3(m^3-1)^3}{I^3}=\frac{(8m^6+20m^3-1)^2}{27J^2}.$$

See H. Hilton, *Plane Algebraic Curves*, Oxford, 1920, p. 235.

Substituting the expressions for A, B, C, D, we find that

$$\Theta = y_1^2 z_1^2 X_2^2 X_3^2 [(x_3 y_1 z_2 - x_2 y_3 z_1)^2 - 4\Delta x_1 y_2 z_3 - 4X_1 Y_2 Z_3],$$

$$ABCD = y_1^4 z_1^4 X_2^4 X_3^4 [x_1 X_1 y_2 Y_2 z_3 Z_3 \Delta].$$

The nature of the cubic is thus seen to depend on the value of the single invariant

$$K = \frac{[(x_3 y_1 z_2 - x_2 y_3 z_1)^2 - 4\Delta x_1 y_2 z_3 - 4X_1 Y_2 Z_3]^2}{x_1 y_2 z_3 X_1 Y_2 Z_3 \Delta}.$$

The cubic has a double point when this invariant is either infinite or 64. When $\Theta = 0$ we have $J = 0$ and the four tangents form a harmonic pencil. If A, B, C, Y, and Z are given, the locus of X when $\Theta = 0$ is a conic. This, however, is not the complete locus when the tangents form a harmonic pencil because J vanishes also when $\Theta^2 = 72ABCD$ or $K = 72$. It should be noticed that the expression K remains the same when we replace each constituent in the determinant by its co-factor. We have in fact an identity of type

$$X_3 Y_1 Z_2 - X_2 Y_3 Z_1 = \Delta(z_3 Z_3 - y_2 Y_2 + x_3 X_3 - x_2 X_2)$$
$$= \Delta(x_3 y_1 z_2 - x_2 y_3 z_1),$$

with the aid of which the preceding remark is easily verified.

Quantities that possess this property may be called *invariants* of the determinant. To find such quantities we may commence by finding pseudo-invariants, that is, quantities which retain the same form except for a power of the determinant. The quantities $x_1 X_1$, $x_2 X_2$, $x_3 X_3$, $y_1 Y_1$, $y_2 Y_2$, $y_3 Y_3$, $z_1 Z_1$, $z_2 Z_2$, $z_3 Z_3$ all possess this last property when considered as functions of the constituents x_1, x_2, etc. and a similar remark may be made in the case of a determinant of higher order. For the general case of a determinant of order n I do not know definitely how many pseudo-invariants of degree n in the constituents are linearly independent. A consideration of the cases $n = 2$, 3, and 4 leads to the conclusion that this number N may be given by the formula

$$N = (n - 1)^2 + 1.$$

When $n=2$, it is evident that $N=2$. When $n=3$ it appears that $N=5$, for each of the nine quantities x_1X_1, x_2X_2, x_3X_3, y_1Y_1, y_2Y_2, y_3Y_3, z_1Z_1, z_2Z_2, z_3Z_3 can be expressed as a linear function of the five linearly independent quantities x_1X_1, x_2X_2, y_1Y_1, y_2Y_2, Δ.

In the case of a determinant of the fourth order

$$\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \\ t_1 & t_2 & t_3 & t_4 \end{vmatrix},$$

the problem is more complex because in addition to the pseudo-invariants x_sX_s etc. there are pseudo-invariants of type $(y_2z_3-y_3z_2)(x_1t_4-x_4t_1)$.

Adopting the notation

$$\binom{yz}{14} = (y_2z_3 - y_3z_2)(x_1t_4 - x_4t_1) = \binom{xt}{23},$$

we observe that the relations of type

$$y_1Y_1 = x_2X_2 + \binom{xt}{12} + \binom{zx}{12} - \binom{xy}{31} - \binom{xy}{14},$$

$$y_2Y_2 = x_1X_1 + \binom{zx}{12} + \binom{xt}{12} + \binom{xy}{24} + \binom{xy}{23},$$

$$y_3Y_3 = x_4X_4 + \binom{xt}{34} + \binom{zx}{34} - \binom{xy}{31} - \binom{xy}{23},$$

$$y_4Y_4 = x_3X_3 + \binom{xt}{34} + \binom{zx}{34} - \binom{xy}{14} - \binom{xy}{24}$$

indicate that all the pseudo-invariants of type y_sY_s, z_sZ_s, t_sT_s can be expressed in terms of the pseudo-invariants x_1X_1, x_2X_2, x_3X_3, x_4X_4 and the 18 quantities of type

$$\binom{yz}{14}.$$

It would be wrong, however, to suppose that there are 22 linearly independent pseudo-invariants of degree four, for there are the six identities

$$x_1X_1 + x_4X_4 = \binom{xy}{23} + \binom{zx}{23} + \binom{xf}{23},$$

$$x_2X_2 + x_4X_4 = \binom{xy}{31} + \binom{zx}{31} + \binom{xf}{31},$$

$$x_3X_3 + x_4X_4 = \binom{xy}{12} + \binom{zx}{12} + \binom{xf}{12},$$

$$x_2X_2 + x_3X_3 = \binom{xy}{14} + \binom{zx}{14} + \binom{xf}{14},$$

$$x_3X_3 + x_1X_1 = \binom{xy}{24} + \binom{zx}{24} + \binom{xf}{24},$$

$$x_1X_1 + x_2X_2 = \binom{xy}{34} + \binom{zx}{34} + \binom{xf}{34},$$

three relations of type

$$\Delta = \binom{xy}{34} + \binom{xy}{14} + \binom{xy}{24} + \binom{xy}{23} + \binom{xy}{31} + \binom{xy}{12},$$

three relations of type

$$\Delta = \binom{yz}{12} + \binom{zx}{12} + \binom{xy}{12} + \binom{xf}{12} + \binom{yf}{12} + \binom{zf}{12},$$

and the relation $x_1X_1 + x_2X_2 + x_3X_3 + x_4X_4 = \Delta$. The number of linearly independent pseudo-invariants thus seems to be ten.

California Institute of Technology

ON THE INTEGRATION IN FINITE TERMS OF LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER*

BY J. F. RITT

1. *Introduction.* Liouville, in 1840, investigated the equation

$$\text{(A)} \qquad \frac{d^2w}{dz^2} = \chi(z)w,$$

with $\chi(z)$ an elementary function of z, to determine under what circumstances the solutions of the equation are elementary.†

By an "elementary function", Liouville understood, in this connection, any function of z obtained in a finite number of steps by performing algebraic operations, taking logarithms and exponentials, and performing integrations. An example of such a function would be

$$\int\left[\log \arcsin z + \int z^2 dz\right]^{1/2} dz + \tan \log_z[1 + (z)^{1/2}].$$

It is a consequence of Liouville's work that the solutions of Bessel's equation

$$z^2w'' + zw' + (z^2 - \nu^2)w = 0$$

are elementary functions only when 2ν is an odd integer.‡ But it would be improper to conclude from Liouville's results alone, as some authors seem to do, that Bessel's equation cannot be solved by the simpler formal methods of the theory of differential equations. For instance, the equation

$$\frac{du}{dz} = \frac{u}{u - 1}$$

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* Presented to the Society, October 25, 1924.

† Journal de Mathématiques, vol. 5 (1840). See also Watson, *Theory of Bessel Functions*, Cambridge, 1922, p. 111. No acquaintance with Liouville's work is necessary for the understanding of the present paper.

‡ Watson, loc. cit.

integrates directly into

$$u - \log u = z + c,$$

but the functions u thus obtained cannot be expressed in terms of z by performing algebraic operations and taking exponentials and logarithms. Can one be sure in advance that Bessel's equation is not susceptible to similar treatment?

Investigations on the possibility of integrating differential equations of the first order by elementary operations performed upon both the dependent variable and the independent variable were carried out by Lorenz, Steen, Hansen and Petersen,* and by Mordukhai-Boltovskoi.† Lagutinski has studied systems of equations of the first order.‡

In the present paper, we prove a general theorem (§3) which shows that the solutions of Bessel's equation do not satisfy elementary equations,§ except for the given values of ν.

Our theorem states that if any solution of equation (A) satisfies an elementary equation in w and z, then the general solution of (A) is an elementary function of z.

The procedure in the present paper is purely formal. The many questions of a function-theoretic nature which will arise will be found treated in our papers on elementary functions published in the Transactions of this Society.‖

2. *The l-Functions.* We shall call any algebraic function of w and z an *l-function of order zero*, and the variables w and z *monomials of order zero.*

We introduce now two classes of *monomials of order one.* The first class will consist of the exponentials of non-constant algebraic functions of w and z. The second will consist of all

* Nyt Tidskrift for Matematik, 1874–1876.

† Communications de la Société Mathématique de Kharkow, vol. 10 (1909), p. 34.

‡ Kharkow Technological Institute, 1909. I have not been able to secure Lagutinski's paper, but the abstract in the Jahrbuch indicates that the present paper does not overlap on it.

§ This term, whose meaning is fairly clear from what precedes, is made precise below.

‖ Vol. 25 (1923), p. 211, and vol. 27 (1925), p. 68.

functions which are not algebraic, but whose partial derivatives are both algebraic. Evidently the logarithm of any non-constant algebraic function is a monomial of the second class.

By an *l-function of order one*, we shall mean any non-algebraic function u of w and z which is obtained by performing algebraic operations upon monomials of orders zero and one; that is, a function which satisfies an equation

$$(1) \qquad \alpha_0(w,z)u^n + \alpha_1(w,z)u^{n-1} + \cdots + \alpha_n(w,z) = 0,$$

in which every $\alpha(w, z)$ is a rational integral combination of monomials of orders zero and one.

We now define, by induction, *l*-functions of any order n. A function which is not among the functions of order 0, 1, $\cdots$, $n-1$, will be called a *monomial of order n* if it is of one of the following two types:

(a) an exponential of a function of order $n-1$;

(b) a function both of whose partial derivatives are among the functions of order 0, 1, $\cdots$, $n-1$.

Evidently a logarithm of a function of order $n-1$ is a monomial of order n if it is not a function of order less than n.

Any function u will be called an *l-function of order n* if it is not an *l*-function of order less than n, and if it satisfies an equation like (1) in which each $\alpha(w, z)$ is a rational integral combination of monomials of orders 0, 1, $\cdots$, n.*

The result of performing algebraic operations upon *l*-functions of the first n orders is always an *l*-function of one of those orders.

The partial derivatives of any *l*-function u of order n are both *l*-functions of order n or less. If, in the expression for u, the maximum of the orders of the monomials which involve w is r, then the derivatives have expressions for which the same maximum is at most r, and in which no monomials of order r involving w appear which do not also appear in u. These facts are a consequence of the rules for differentiation.

* It is not important here to know whether functions of every order actually exist. See Liouville, Journal de Mathématiques, vol. 3 (1838), p. 523.

3. *Linear Differential Equations.* We prove the following theorem, which shows that the solutions of Bessel's equation do not satisfy an elementary equation, except for the stated values of ν.

THEOREM. *If w, any solution of the differential equation $w''=\chi(z)w$, in which $\chi(z)$ is an l-function, satisfies an equation $F(w, z)=0$, where $F(w, z)$ is an l-function, not identically zero, then w is an l-function of z.*

Differentiating the equation $F(w, z)=0$ with respect to z, we find

$$F_w w' + F_z = 0. \tag{2}$$

Here F_w and F_z are l-functions. We have thus, in the first member of (2), an expression algebraic in w', and in l-functions of w and z, which vanishes when w is the given solution of $w''=\chi(z)w$, *without vanishing identically.**

With every such expression, $f(w', w, z)$, we associate two integers; first r, the maximum of the orders of those monomials in w and z, contained in the expression, which involve w; secondly s, the number of such monomials of order r which involve w. Many different expressions will represent the same function; the integers will vary with the expression.

There is a class of expressions $f(w', w, z)$ for which r is a minimum, and in this class there are expressions for which s is as small as it can be if r is a minimum. We assume that we have in hand an $f(w', w, z)$ with r and s minima, and write

$$f(w', w, z) = 0. \tag{3}$$

We are going to prove that $r=0$. Let the contrary, namely that $r>0$, be assumed, and let θ be one of the monomials of order r contained in f which involve w. Since f involves θ algebraically, we find from (3), upon solving for θ,

$$\theta - g(w', w, z) = 0, \tag{4}$$

* Of course, $F(w, z)$ is also such an expression, but (2) gives a better example.

where g is an algebraic function of w' and of l-functions of w and z, with $s-1$ monomials of order r which involve w.

If θ is an exponential, e^v, where v is of order $r-1$, we write (4) in the form

$$v - \log g(w', w, z) = 0. \tag{5}$$

If θ is an integral, that is, a monomial of the second class, we let (4) stand. We let the first member of the equation

$$G(w', w, z) = 0 \tag{6}$$

represent the first member of (4), or that of (5), according as θ is an integral or an exponential.

On differentiating (6) we find, remembering that $w'' = \chi(z)w$,

$$\chi(z)wG_{w'} + G_w w' + G_z = 0. \tag{7}$$

Now if the first member of (7) did not vanish identically with respect to w', w and z, we would have an equation of the form (3) involving at most $s-1$ monomials of order r which contain w. For, in (4) and (5), the partial derivatives of θ and v are of order less than r, and the derivatives of g and $\log g$ contain at most $s-1$ monomials of order r involving w.* Hence the first member of (7) vanishes identically.

Let μ be any constant. If w is the given solution of the equation $w'' = \chi(z)w$, the derivative with respect to z of

$$G(\mu w', \mu w, z) \tag{8}$$

will evidently be the first member of (7) with w' and w replaced by $\mu w'$ and μw.† As (7) holds for arbitrary values of the variables, the derivative just mentioned must vanish for every z. Hence the function of (8) depends only on μ, and not on z. We write

$$G(\mu w', \mu w, z) = \beta(\mu). \tag{9}$$

* Final remarks of §2, which obviously apply also to $G_{w'}$.

† The device now being used is analogous to, though not identical with, one which underlies Liouville's work on the elementary functions. We use below a method given by us in the Comptes Rendus for Aug. 21, 1926, which permits a great simplification of the work of Liouville and his followers.

We differentiate (9) partially with respect to μ, and then put $\mu=1$. Writing $\beta'(1)=a$, we have

$$w'G_{w'} + wG_w = a, \tag{10}$$

and, as in the case of (7), we see that (10) is an identity in w', w and z.

A particular solution of (10) is $G=a \log w$, and w'/w is a solution of the equation obtained on replacing the second member of (10) by zero. Hence

$$G = a \log w + H\left(\frac{w'}{w}, z\right). \tag{11}$$

As G is algebraic in w', H must be an algebraic function of w'/w and of l-functions of z. By (6) and (11), we have, for w as in the hypothesis,

$$a \log w + H\left(\frac{w'}{w}, z\right) = 0. \tag{12}$$

If $a=0$, we have a contradiction of the assumption that $r>0$.

Suppose that $a\neq 0$. Writing $u=w'/w$ and differentiating (12), we find

$$au + [\chi(z) - u^2]H_u + H_z = 0. \tag{13}$$

This equation must be an identity in u and z, else it would be an equation (3) with $r=0$. But we shall show that no H which involves u algebraically can satisfy (13), thus making untenable, finally, the assumption that $r>0$.

If H is algebraic in u, any of its branches has, for the neighborhood of $u=\infty$, a development in descending powers of u,

$$H = \alpha_1(z)u^{p_1} + \alpha_2(z)u^{p_2} + \cdots + \alpha_n(z)u^{p_n} + \cdots, \tag{14}$$

where the decreasing exponents p_n are rational, and all have the same denominator, which may be unity. We may, and shall, assume that no $\alpha(z)$ is zero for every z.

If $p_1\neq 0$, the development of $(\chi-u^2)H_u$ begins with a term in u^{p_1+1}, whereas, in H_z, the first exponent does not exceed p_1. Hence the first term of $(\chi-u^2)\,H_u$ must balance with au

in (13), an impossibility if $p_1 \neq 0$. If $p_1 = 0$, $(\chi - u^2)H_u$ begins with u^{p_2+1}, and H_z begins with a zero or negative power. Now, as $a \neq 0$, we have to balance au in (13). This is impossible, because $p_2 + 1 < 1$. Thus no H algebraic in u satisfies (13).

We have proved that $r = 0$ for the first member of (3). If, now, (3) does not involve w', then, since it is algebraic in w, we can solve for w, finding w to be an l-function of z. If (3) does involve w', we solve for w', finding

$$w' = H(w,z), \tag{15}$$

where H is an algebraic function of w, and of l-functions of z.

On differentiating (15), remembering that $w'' = \chi(z)w$, we find

$$\chi(z)w = H_w H + H_z. \tag{16}$$

If (16) is not an identity in w and z, w is determined as an l-function of z. Let, then, (16) be an identity, and consider, for the neighborhood of $w = \infty$, a development of H in descending powers,

$$H = \alpha_1(z)w^{p_1} + \alpha_2(z)w^{p_2} + \cdots + \alpha_n(z)w^{p_n} + \cdots .$$

As the coefficients α are found by algebraic operations from the equation which H satisfies together with w and l-functions of z, the α's are all l-functions.

When we substitute the development of H into (16), we find that $p_1 = 1$ and that

$$\alpha_1'(z) + [\alpha_1(z)]^2 = \chi(z).$$

Thus, $\alpha_1(z)$ is a solution of the Riccati equation $u' + u^2 = \chi(z)$, obtained from $w'' = \chi(z)w$ by putting $u = w'/w$. Hence the exponential of the integral of $\alpha_1(z)$, which is an l-function not identically zero, satisfies the equation $w'' = \chi(z)w$.

But it is well known, and easy to see, that when a single solution of a linear homogeneous equation of the second order is known, the general solution can be found from it by a quadrature. Hence every solution of (A) is an l-function, and the theorem is proved.

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ON SMALL DEFORMATIONS OF CURVES

BY C. E. WEATHERBURN

1. *Introduction.* This paper is concerned with small deformations of a single tortuous curve, of a family of curves on a given surface, and of a congruence of curves in space. In all cases, the displacement $\boldsymbol{s}$ is supposed to be a small quantity of the first order, quantities of higher order being negligible.*

2. *Single Twisted Curve.* Consider first a given curve in space. The position vector $\boldsymbol{r}$ of a point on the curve may be regarded as a function of the arc-length s of the curve, measured from a fixed point on it. Let $\boldsymbol{t}$, $\boldsymbol{n}$, $\boldsymbol{b}$ be the unit tangent, principal normal and binormal. These are connected with the curvature κ and the torsion τ as in the Serret-Frenet formulas. Imagine a small deformation of the curve, such that the point of the curve originally at $\boldsymbol{r}$ suffers a small displacement $\boldsymbol{s}$, its new position vector $\boldsymbol{r}_1$ being then

$$\boldsymbol{r}_1 = \boldsymbol{r} + \boldsymbol{s}. \tag{1}$$

Let a suffix unity be used to distinguish quantities belonging to the deformed curve, and let primes denote differentiations with respect to the arc-length s. Then the element $d\boldsymbol{r}_1$ of the deformed curve, corresponding to the element $d\boldsymbol{r}$ of the original, is given by $d\boldsymbol{r}_1 = d\boldsymbol{r} + d\boldsymbol{s}$, and its length ds_1 by

$$(ds_1)^2 = (d\boldsymbol{r}_1)^2 = (d\boldsymbol{r})^2 + 2d\boldsymbol{r} \cdot d\boldsymbol{s} = ds^2(1 + 2\boldsymbol{t} \cdot \boldsymbol{s}').$$

Consequently $ds_1 = ds(1 + \boldsymbol{t} \cdot \boldsymbol{s}')$.

The quantity $\boldsymbol{t} \cdot \boldsymbol{s}'$ represents the increase of length per unit length of the curve, or the extension of the curve at

*See also a paper by Perna, Giornale di Matematiche, vol. 36 (1898), pp. 286–299; and another by Salkowski, Mathematische Annalen, vol. 66 (1908), pp. 517-557.

the point considered. Let it be denoted by ϵ. Then

$$ds_1 = ds(1 + \epsilon)\ ; \quad ds = ds_1(1 - \epsilon). \tag{2}$$

The unit tangent t_1 to the deformed curve is given by

$$t_1 = \frac{dr_1}{ds_1} = (1 - \epsilon)(r' + s') = (1 - \epsilon)t + s'. \tag{3}$$

Consequently, if κ_1 is the curvature and n_1 the unit principal normal for the new curve,

$$\kappa_1 n_1 = \frac{dt_1}{ds_1} = (1 - \epsilon)\frac{dt_1}{ds} = (1 - 2\epsilon)\kappa n - \epsilon' t + s''.$$

On "squaring" both members, and neglecting small quantities of the second order, we have

$$\kappa_1^2 = (1 - 4\epsilon)\kappa^2 + 2\kappa n \cdot s'',$$

and therefore

$$\kappa_1 = (1 - 2\epsilon)\kappa + n \cdot s''. \tag{4}$$

Inserting this value in the above product $\kappa_1 n_1$ we find

$$\left\{\begin{aligned} n_1 &= n - \frac{1}{\kappa}((n \cdot s'')n + \epsilon' t - s'') \\ &= n - (n \cdot s')t + \frac{1}{\kappa}(b \cdot s'')b. \end{aligned}\right. \tag{5}$$

The unit binormal b to the deformed curve is then given by

$$\left\{\begin{aligned} b_1 &= t_1 \times n_1 = (1 - \epsilon)b + s' \times n - \frac{1}{\kappa}(b \cdot s'')n \\ &= b - (b \cdot s')t - \frac{1}{\kappa}(b \cdot s'')n. \end{aligned}\right. \tag{6}$$

The torsion τ_1 may then be found by differentiating the unit binormal, using the Serret-Frenet formulas. Thus on differentiating (6), and using the preceding results, we find on reduction

$$\tau_1 = (1 - \epsilon)\tau + \kappa b \cdot s' + \frac{d}{ds}\left(\frac{1}{\kappa} b \cdot s''\right). \tag{7}$$

We have thus determined the geometric characteristics of the deformed curve in terms of those of the original curve and the small displacement $\mathbf{s}$. The rectangular trihedron, consisting of the unit tangent, the principal normal, and the binormal, undergoes a small rotation which is represented by the vector

$$\boldsymbol{R} = \frac{1}{\kappa}(\boldsymbol{b}\cdot\boldsymbol{s}'')\boldsymbol{t} - (\boldsymbol{b}\cdot\boldsymbol{s}')\boldsymbol{n} + (\boldsymbol{n}\cdot\boldsymbol{s}')\boldsymbol{b} \tag{8}$$

or

$$\boldsymbol{R} = \frac{1}{\kappa}(\boldsymbol{b}\cdot\boldsymbol{s}'')\boldsymbol{t} + \boldsymbol{t}\times\boldsymbol{s}'. \tag{9}$$

The coefficients of $\boldsymbol{t}$, $\boldsymbol{n}$, $\boldsymbol{b}$ in (8) represent the small rotations of the trihedron about the tangent, the principal normal, and the binormal, respectively.

An *inextensional deformation* of the curve is one for which ϵ vanishes identically.* If $\mathbf{s}$ is expressed in the form

$$\mathbf{s} = P\boldsymbol{t} + Q\boldsymbol{n} + R\boldsymbol{b},$$

the vanishing of $\boldsymbol{t}\cdot\mathbf{s}'$ gives $P' = \kappa Q$ as the necessary and sufficient condition for inextensional deformation.

3. *Family of Curves on a Surface.* Consider next a family of curves on a given surface. Suppose that the curves of the family suffer a small deformation such that they remain on the same surface. Then the displacement $\mathbf{s}$ at any point is tangential to the surface, and is a function of two parameters that specify the point considered. On any one curve $\mathbf{s}$ is a function of the arc-length s, and the formulas found above hold for the deformed curve.

Other results may be very neatly expressed in terms of the two-parametric differential invariants for the surface, introduced and examined by the author in a recent paper *On differential invariants in geometry of surfaces.*† If ϕ is the value of any function associated with the deformed curve at the point $\boldsymbol{r}+\mathbf{s}$, the new value of this function at

*This case has been considered in some detail by Sannia, Rendiconti di Palermo, vol. 21 (1906), pp. 229–256.

† Quarterly Journal, vol. 50 (1925), pp. 230–269.

the point $\boldsymbol{r}$ originally occupied by this point of the curve is $\boldsymbol{\phi}-\boldsymbol{s}\cdot\nabla\boldsymbol{\phi}$, where ∇ is the two-parametric differential operator for the surface. Thus after the deformation the unit tangent $\bar{\boldsymbol{t}}$ to the new curve through the point $\boldsymbol{r}$ is, by (3),

$$(10)\qquad \begin{cases} \bar{\boldsymbol{t}} = \boldsymbol{t}_1 - \boldsymbol{s}\cdot\nabla\boldsymbol{t}_1 = (1-\epsilon)\boldsymbol{t} + \boldsymbol{t}\cdot\nabla\boldsymbol{s} - \boldsymbol{s}\cdot\nabla\boldsymbol{t} \\ \quad = (1-\epsilon)\boldsymbol{t} + \operatorname{curl}(\boldsymbol{s}\times\boldsymbol{t}) - \boldsymbol{s}\operatorname{div}\boldsymbol{t} + \boldsymbol{t}\operatorname{div}\boldsymbol{s}. \end{cases}$$

We have shown elsewhere* that the line of striction of a family of curves with unit tangent $\boldsymbol{t}$ is given by div $\boldsymbol{t}=0$. Hence the line of striction of the deformed family has for its equation div $\bar{\boldsymbol{t}}=0$, which may be expressed in the form

$$(11)\qquad (1-\epsilon)\operatorname{div}\boldsymbol{t} - \boldsymbol{t}\cdot\nabla\epsilon - \boldsymbol{s}\cdot\nabla\operatorname{div}\boldsymbol{t} + \boldsymbol{t}\cdot\nabla\operatorname{div}\boldsymbol{s} = 0,$$

since the divergence of the curl of the normal vector $\boldsymbol{s}\times\boldsymbol{t}$ vanishes identically.†

If the original family of curves is one of parallels, div $\boldsymbol{t}$ vanishes identically.‡ Hence *a family of parallels will remain parallels after the deformation provided*

$$(12)\qquad \boldsymbol{t}\cdot\nabla(\epsilon - \operatorname{div}\boldsymbol{s}) = 0.$$

The geodesic curvature of a curve of the original family§ is $\boldsymbol{n}\cdot\operatorname{curl}\boldsymbol{t}$, and that of one of the deformed curves is

$$\boldsymbol{n}\cdot\operatorname{curl}[(1-\epsilon)\boldsymbol{t} + \boldsymbol{t}\cdot\nabla\boldsymbol{s} - \boldsymbol{s}\cdot\nabla\boldsymbol{t}].$$

If then the family of curves is a family of geodesics, it will remain so after the deformation provided

$$\boldsymbol{n}\cdot[\boldsymbol{t}\times\nabla\epsilon + \operatorname{curl}(\boldsymbol{t}\cdot\nabla\boldsymbol{s} - \boldsymbol{s}\cdot\nabla\boldsymbol{t})] = 0.$$

Similarly, the original family will be *lines of curvature*‖ if $\boldsymbol{t}\cdot\operatorname{curl}\boldsymbol{t}=0$; and they will remain lines of curvature after the change provided $\bar{\boldsymbol{t}}\cdot\operatorname{curl}\bar{\boldsymbol{t}} = 0$.

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* *Some new theorems in geometry of a surface*, § 2, Mathematical Gazette, vol. 13 (1926), pp. 1–6.

† Quarterly Journal, loc. cit., § 7.

‡ See a paper by the author *On families of curves and surfaces*, § 7, recently communicated to the Messenger of Mathematics.

§ Quarterly Journal, loc. cit., § 8.

‖ Mathematical Gazette, loc. cit., § 4.

4. *Congruence of Curves in Space.* Consider finally a small deformation of a congruence of curves in three dimensions. On any one curve the displacement $\mathbf{s}$ is a function of a single parameter; but on the congruence it is a point-function in space. Let ∇ now represent the three-parametric differential operator for three dimensions. Then the extension ϵ is $\epsilon = \mathbf{t} \cdot \mathbf{s}' = \mathbf{t} \cdot (\nabla \mathbf{s}) \cdot \mathbf{t}$, and, by the same argument as above, the unit tangent $\bar{\mathbf{t}}$ to the curve of the congruence which passes through the point $\mathbf{r}$ after the deformation is $\bar{\mathbf{t}} = (1 - \epsilon)\mathbf{t} + \mathbf{t} \cdot \nabla \mathbf{s} - \mathbf{s} \cdot \nabla \mathbf{t}$, which may also be expressed in the same form as (10).

The surface of striction (or orthocentric surface) of a congruence with unit tangent $\mathbf{t}$ is given by* $\operatorname{div} \mathbf{t} = 0$. Hence the surface of striction of the deformed congruence has for its equation $\operatorname{div} \bar{\mathbf{t}} = 0$, which may be expanded in the form $(1 - \epsilon) \operatorname{div} \mathbf{t} - \mathbf{t} \cdot \nabla \epsilon - \mathbf{s} \cdot \nabla \operatorname{div} \mathbf{t} + \mathbf{t} \cdot \nabla \operatorname{div} \mathbf{s} = 0$, since the divergence of the curl now vanishes identically. The limit surface† of the congruence before deformation is

$$\operatorname{div} (\mathbf{t} \operatorname{div} \mathbf{t} - \mathbf{t} \cdot \nabla \mathbf{t}) = 0, \dagger \tag{13}$$

and that of the deformed congruence is given by a similar equation in which $\bar{\mathbf{t}}$ takes the place of $\mathbf{t}$. The deformed curves will constitute a normal congruence provided we have

$$\bar{\mathbf{t}} \cdot \operatorname{curl} \bar{\mathbf{t}} = 0,$$

and they will constitute an isometric congruence if, in addition, $\bar{\mathbf{t}}$ satisfies the equation§

$$\operatorname{curl} (\bar{\mathbf{t}} \operatorname{div} \bar{\mathbf{t}} - \bar{\mathbf{t}} \cdot \nabla \bar{\mathbf{t}}) = 0 \tag{14}$$

which may be expanded in terms of $\mathbf{t}$ and $\mathbf{s}$.

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* See a paper by the author *On congruences of curves*, § 6, recently communicated to the Tôhoku Mathematical Journal.

† Ibid., § 4.

‡ This equation for the limit surface of a curvilinear congruence was first given by the author in a paper *On isometric systems of curves and surfaces*, recently communicated to the American Journal.

§ Ibid., § 6.

INTEGERS REPRESENTED BY POSITIVE TERNARY QUADRATIC FORMS*

BY L. E. DICKSON

1. *Introduction.* Dirichlet† proved by the method of §2 the following two theorems:

THEOREM I. *$A=x^2+y^2+z^2$ represents exclusively all positive integers not of the form $4^k(8n+7)$.*

THEOREM II. *$B=x^2+y^2+3z^2$ represents every positive integer not divisible by 3.*

Without giving any details, he stated that like considerations applied to the representation of multiples of 3 by B. But the latter problem is much more difficult and no treatment has since been published; it is solved below by two methods.

Ramanujan‡ readily found all sets of positive integers a, b, c, d such that every positive integer can be expressed in the form $ax^2+by^2+cz^2+du^2$. He made use of the forms of numbers representable by

$$\begin{aligned} A, B, C &= x^2+y^2+2z^2, \\ D &= x^2+2y^2+2z^2, \\ E &= x^2+2y^2+3z^2, \\ F &= x^2+2y^2+4z^2, \\ G &= x^2+2y^2+5z^2. \end{aligned}$$

He gave no proofs for these forms and doubtless obtained his results empirically. We shall give a complete theory for these forms. These cases indicate clearly methods of procedure for any similar form.

For a new theorem on forms in n variables, see §9.

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* Presented to the Society, December 31, 1926.

† Journal für Mathematik, vol. 40 (1850), pp. 228–32; French translation Journal de Mathématiques, (2), vol. 4 (1859), pp. 233–40; Werke, vol. II, pp. 89–96.

‡ Proceedings of the Cambridge Philosophical Society, vol. 19 (1916–19), pp. 11–15. He overlooked the fact that $x^2+2y^2+5z^2+5u^2$ fails to represent 15.

2. *The Form B.* Let B represent a multiple of 3. Since -1 is a quadratic non-residue of 3, x and y must be multiples of 3. Thus $B=3\beta$, $\beta=3X^2+3Y^2+z^2$. Since $\beta\equiv 0$ or $1 \pmod 3$, β represents no integer $3n+2$. If β is divisible by 3, z is divisible by 3 and B is the product of a like form by 9. We shall prove that β represents every positive integer $3n+1$. These results and Theorem II give

THEOREM III. *$x^2+y^2+3z^2$ represents exclusively all positive integers not of the form $9^k(9n+6)$.*

We shall change the notation from β to f and employ the fact that the only reduced positive ternary forms of Hessian 9 are*

$$f = x^2 + 3y^2 + 3z^2, \qquad g = x^2 + y^2 + 9z^2,$$
$$h = x^2 + 2y^2 + 5z^2 - 2yz, \qquad l = 2x^2 + 2y^2 + 3z^2 - 2xy.$$

No one of g, h, l represents an integer $8m+7$. For g this follows from Theorem I, since $g\equiv A \pmod 8$. Suppose $l\equiv 7 \pmod 8$. Then z is odd, $2s\equiv 4 \pmod 8$, where $s=x^2+y^2-xy$. Thus s is even and $(1+x)(1+y)\equiv 1 \pmod 2$, x and y are even, and $s\equiv 0 \pmod 4$, a contradiction. Finally, let $h\equiv 7 \pmod 8$. If y is even, $h\equiv x^2+z^2 \pmod 4$. Hence y is odd and

$$3\equiv h\equiv x^2+(z-1)^2+1 \pmod 4,$$

so that x and $z-1$ are odd. Write $z=2Z$. Then $h\equiv 3+4Z(Z-1)\equiv 3 \pmod 8$.

Consider the ternary form lacking the term xy:

$$\phi = ax^2 + by^2 + cz^2 + 2ryz + 2sxz. \tag{1}$$

Its Hessian H is $a\Delta-bs^2$, where $\Delta=bc-r^2$. Take $H=9$, $s=1$, $\Delta=24t$, $t=6k+1$. Then $b=3\beta$, $\beta=8at-3$. If a is not divisible by 3, $\beta=48ak+8a-3$ is a linear function of k with relatively prime coefficients and hence represents an infinitude of primes.

* Eisenstein, Journal für Mathematik, vol. 41 (1851), p. 169. By the Hessian H of ϕ we mean the determinant whose elements are the halves of the second partial derivatives of ϕ with respect to x, y, z. Eisenstein called $-H$ the determinant of ϕ. The facts borrowed in this paper from Eisenstein's table have been verified independently by the writer.

Take $a=3n+1$. Then $\beta\equiv -1 \pmod 6$,

$$\left(\frac{-3}{\beta}\right) = -1, \quad \left(\frac{2}{\beta}\right) = -1,$$

$$\left(\frac{t}{\beta}\right) = \left(\frac{\beta}{t}\right) = \left(\frac{-3}{t}\right) = 1, \quad \left(\frac{-\Delta}{\beta}\right) = 1. \tag{2}$$

Hence $w^2\equiv -\Delta \pmod \beta$ is solvable. We can choose a multiple r of 3 such that $r\equiv w \pmod \beta$. Then $(\Delta+r^2)/b$ is an integer c. Since ϕ represents $b\equiv 7 \pmod 8$, it is equivalent to no one of g, h, l and hence is equivalent to f. Thus f represents every $a=3n+1$.

THEOREM IV. *$x^3+3y^2+3z^2$ represents exclusively all positive integers not of the form $9^k(3n+2)$.*

This theory for B made use of forms of the larger Hessian 9. We shall next show how to deduce a theory making use only of forms having the same Hessian 3 as B.

3. *A New Theory for B.* We proved that f is equivalent to a form (1) having $a=3n+1$, $b=3\beta$, $r=3\rho$, $s=1$, where $\beta=8at-3$ is a prime. In $9=H=a(bc-r^2)-3\beta$, replace β by its value. Thence $c=(8t+3\rho^2)/\beta\equiv 1 \pmod 3$, $c=1+3\gamma$. In (1) replace x by $X-z$. We get

$$\psi = aX^2 - 6nXz + 3(n+\gamma)z^2 + 3\beta y^2 + 6\rho yz.$$

Write $3z=Z$, $3y=Y$. Then

$$3\psi = 3aX^2 - 6nXZ + (n+\gamma)Z^2 + \beta Y^2 + 2\rho YZ$$

is equivalent to $3f=3x^2+Y^2+Z^2$ and hence to B. In 3ψ, replace a by α, β by b, ρ by r. We conclude that (1) is equivalent to B if

$$a = 3\alpha, \quad \alpha = 3n+1, \quad b = 8\alpha t - 3,$$
$$t = 6k+1, \quad s = -3n = 1-\alpha. \tag{3}$$

We shall now give a direct proof that there exists a form (1) of Hessian 3 which satisfies conditions (3) and is equivalent to B. In $H=3$, replace a and s by their values in (3). We get

$$b + 3 + 3\alpha r^2 - \alpha bP = 0, \quad P = 3c + 2 - \alpha.$$

Replace the first term b by its value in (3), and cancel α. We get

$$(4) \qquad 8t + 3r^2 - bP = 0, \quad -24t \equiv (3r)^2 \pmod{b}.$$

This congruence is solvable by (2) with β replaced by b. By (4), $8t(1-P) \equiv 0$, $P \equiv 1 \pmod 3$. Hence the value of c determined by P is an integer. The only two reduced forms of Hessian 3 are B and $\chi = x^2+2\sigma$, where $\sigma = y^2+yz+z^2$. Suppose $\chi \equiv 5 \pmod 8$. Then x is odd and $\sigma \equiv 2 \pmod 4$. Thus

$$(1+y)(1+z) \equiv 1, \quad y \equiv z \equiv 0 \pmod 2, \quad \sigma \equiv 0 \pmod 4.$$

This contradiction shows that b is not represented by χ. Since (1) represents b, it is not equivalent to χ and hence is equivalent to B. Thus B as well as ϕ represents* $a = 3\alpha$. This completes the new proof of Theorem III by using only forms of Hessian 3. The numbers represented by χ are given by Theorem XI.

4. *The Form* $C = x^2+y^2+2z^2$. By Theorem I, A represents every positive $4k+2$. Then just two of x, y, z are odd, say x and y, while $z = 2Z$. Then $x = X+Y$, $y = X-Y$ determine integers X and Y. Hence

$$X^2+Y^2+2Z^2 = 2k+1,$$

so that C represents all positive odd integers.† If

$$m \neq 4^k(8n+7), m = X^2+Y^2+z^2,$$

by Theorem I. Hence

$$2m = (X+Y)^2+(X-Y)^2+2z^2.$$

Conversely, if C is even, it is of the latter form.

THEOREM V. *C represents exclusively all positive integers not of the form* $4^k(16n+14)$.

5. *The Form* $D = x^2+2y^2+2z^2$. If m is odd and $\neq 8n+7$, $m = x^2+Y^2+Z^2$ by Theorem I. Then $x+Y+Z$ is odd. Permuting, we may take x odd, and write $Y+Z = 2y$, $Y-Z = 2z$.

* If we apply the method of §2 when $H = 3$ and hence take $s = 1$, $b = 3\beta$, where β is a prime $\equiv -1 \pmod 8$, we find that it fails for all choices of Δ.

†Lebesque, Journal de Mathématiques, (2), vol. 2 (1857), p. 149, gave a long proof by the method of §2.

Then $m=D$. Next, let $m=2r$ be any even integer not of the form $4^k(8n+7)$. Then $r\neq 4^t(16n+14)$. By Theorem V, r is represented by C. Then $m=2r$ is represented by D with x even.

THEOREM VI. *D represents exclusively* all positive integers not of the form $4^k(8n+7)$.*

6. *The Form $F=x^2+2y^2+4z^2$.* Every odd integer is represented by C with $x+y$ odd, whence one of x and y is even. Any integer $\neq 4^k(8n+7)$ is represented by D, and $2D=(2y)^2+2x^2+4z^2$.

THEOREM VII. *F represents exclusively† all positive integers not of the form $4^k(16n+14)$.*

The simple methods used in proving Theorems V–VII apply also to $x^2+2^ry+2^sz$ when r and s are both $\leqq 3$, and when $r=1$ or 3, $s=4$.

7. *The Form $G=x^2+2y^2+5z^2$.* The only reduced forms of Hessian 10 are G, $J=x^2+y^2+10z^2$, $K=2x^2+2y^2+3z^2+2xz$, and $L=2x^2+2y^2+4z^2+2yz+2xz+2xy$. Neither J nor K represents a number of the form $2(8n+3)$. For, if K is even, $z=2Z$, $K=2M$, $M=X^2+y^2+5Z^2$, where $X=x+Z$. Since M is congruent to a sum of three squares modulo 4, it is congruent to 3 if and only if each square is odd, and then $M\equiv 7 \pmod 8$. If J is even, $x=y+2t$, $J=2N$, $N=(y+t)^2+t^2+5z^2\not\equiv 3 \pmod 8$.

We now apply the method of §2 to prove that G represents every positive integer prime to 10. Take $\Delta=16k$, $k=10l\pm 3$. Then $b=2\beta$, where $\beta=8ka-5$ represents infinitely many primes. Now

$$\left(\frac{-\beta}{k}\right)=\left(\frac{5}{k}\right)=\left(\frac{k}{5}\right)=-1,$$

$$\left(\frac{-\Delta}{\beta}\right)=-\left(\frac{k}{\beta}\right)=-\left(\frac{-\beta}{k}\right)=1.$$

Since (1) represents b, which is of the form $2(8n+3)$, it is not equivalent to J or K. Since it represents the odd a, it is not

* $D=x^2+(y+z)^2+(y-z)^2\neq 4^k(8n+7)$ by Theorem I.

† If F is even, x is even and F is the double of a form D.

equivalent to L. Hence (1) is equivalent to G, which therefore represents a.

If G represents a multiple of 5, it is the product of 5 by $g=5X^2+10Y^2+z^2$, whence G represents no $5(5n\pm2)$. Also, g is divisible by 5 only when z is. Thus G is divisible by 25 only when it is a product of 25 by a form like G.

To prove* that G represents every 5α if $\alpha=5n\pm1$ is odd, employ (1) with $a=5\alpha$, $b=2\beta$, $\beta=8\alpha t-5$, $r=2\rho$, $s=1\mp\alpha$. The Hessian of (1) is 10 if

$$\beta + 5 + 10\alpha\rho^2 - \alpha\beta P = 0, \qquad P = 5c \pm 2 - \alpha.$$

Take t prime to 10 and replace the first β by its value. Thus $8t+10\rho^2-\beta P=0$, $P\equiv\pm1\pmod 5$. Hence P yields an integral value for c. Also,

$$\left(\frac{t}{\beta}\right) = \left(\frac{-\beta}{t}\right) = \left(\frac{5}{t}\right),$$

$$\left(\frac{5}{\beta}\right) = \left(\frac{\beta}{5}\right) = \left(\frac{\pm 8t}{5}\right) = -\left(\frac{t}{5}\right), \tag{5}$$

$$\left(\frac{-80t}{\beta}\right) = -\left(\frac{5t}{\beta}\right) = 1.$$

Next, if G is even, $x=z+2w$ and $G=2T$, where

$$T = y^2 + 2w^2 + 2wz + 3z^2, \qquad S = x^2 + y^2 + 5z^2$$

are the only reduced forms of Hessian 5. Every positive integer a prime to 5 is represented by T. Take $\Delta=8k, k=10m\pm1$. Then $b=a\Delta-5$ represents an infinitude of primes, and

$$\left(\frac{-2}{b}\right) = 1, \quad \left(\frac{-\Delta}{b}\right) = \left(\frac{k}{b}\right) = \left(\frac{-b}{k}\right) = \left(\frac{5}{k}\right) = \left(\frac{k}{5}\right) = 1.$$

Now $b\equiv3\pmod 8$ is not represented by S, as proved for M. Hence (1) is equivalent to T and not S. Thus T represents a.

* Or we may use the method of §2. Of the ten properly primitive reduced forms of Hessian 50, all except g fail to represent numbers $\equiv14\pmod{16}$. To prove that g represents $\alpha=5n\pm1$ when odd, take $\Delta=80t$, whence $b=10\beta$, $\beta=8\alpha t-5$; apply (5). From this proof was reconstructed the shorter one in the text.

We saw that $2T=G$ represents no $5(5m\pm2)$. Thus T represents no $5(5n\pm1)$. To prove that T represents every 5α, where $\alpha=5n\pm2$, employ (1) with $a=5\alpha$, $s=1\pm2\alpha$. Its Hessian is 5 if

$$b+5+5\alpha r^2-\alpha bP=0, \qquad P=5c-4\alpha\pm4.$$

Replace the first term b by $8t\alpha-5$, where t is prime to 10. Thus $8t+5r^2-bP=0$. Hence $8t(1\mp2P)\equiv0$, $P\equiv\pm3(\text{mod }5)$, and the resulting value of c is an integer. Also

$$\left(\frac{5}{b}\right)=\left(\frac{b}{5}\right)=\left(\frac{\pm16t}{5}\right)=\left(\frac{t}{5}\right),$$

$$\left(\frac{t}{b}\right)=\left(\frac{-b}{t}\right)=\left(\frac{5}{t}\right), \quad \left(\frac{-40t}{b}\right)=1.$$

We have now proved the two theorems:

THEOREM VIII. *T represents exclusively all $\neq25^k(25n\pm5)$.*

THEOREM IX. *G represents exclusively all $\neq25^k(25n\pm10)$.*

8. *The Form $E=x^2+2y^2+3z^2$.* We shall outline a proof of the following theorem.

THEOREM X. *E represents exclusively all $\neq4^k(16n+10)$.*

The only reduced forms of Hessian 6 are E and

$$Q=x^2+y^2+6z^2, \qquad R=2x^2+2y^2+2z^2+2xy.$$

To prove that E represents every positive odd integer a, take $\Delta=9k$, $k=8t+3$. Then $b=3\beta$, where β represents primes. Also $(-\Delta/\beta)=1$. The resulting form (1) represents the odd a and hence is not equivalent to R. Since it represents $b=3(3n+1)$, it is not equivalent to Q. For, if Q is divisible by 3, both x and y are.

If E is even, then $x=z+2t$ and $E=2U$,

$$U=y^2+2z^2+2zt+2t^2$$

and conversely. In place of U we employ the like form χ of §3. To show that χ represents $a=2\alpha$ when α is odd, take $\Delta=9k$, $k=4t+1$. Then $b=a\Delta-3\equiv6(\text{mod }9)$ is not represented by the remaining reduced form B of Hessian 3 (Theorem III). Also $b=3\beta$, $(-\Delta/\beta)=+1$.

Finally, χ represents every positive odd integer $a \not\equiv 5 \pmod 8$. Write $\alpha = \frac{1}{2}(3a-1)$ and take $\Delta = 9k$, $k = 2h+1$. Then $b = 6q$, $q = 3ah+\alpha$. If $a = 8A+1$, take $h = 4t$, t odd. Then $q = 12Ak + 12t + 1$,

$$\left(\frac{-\Delta}{q}\right) = \left(\frac{k}{q}\right) = \left(\frac{q}{k}\right) = \left(\frac{12t+1-k}{k}\right) = \left(\frac{4t}{k}\right) = \left(\frac{k}{t}\right) = 1.$$

If $a = 8A+3$, take $h = 4t+1$. If $a = 8A+7$, take $h = 4t+1$. In each case $(-\Delta/q) = 1$. In all three cases, q represents an infinitude of primes.

THEOREM XI. *$x^2+2y^2+2yz+2z^2$ represents exclusively all positive integers not of the form $4^k(8n+5)$.*

9. *Forms in n Variables.* By a simple modification of Ramanujan's determination of quaternary forms which represent all positive integers, we readily prove*

THEOREM XII. *If, for $n \geqq 5$, $f = a_1x_1^2 + \cdots + a_nx_n^2$ represents all positive integers, while no sum of fewer than n terms of f represents all positive integers, then $n = 5$ and*

$$f = x^2 + 2y^2 + 5z^2 + 5u^2 + ev^2, \quad (e = 5, 11, 12, 13, 14, 15),$$

and these six forms f actually have the property stated.

After this paper was in type, I saw that J. G. A. Arndt gave† the Dirichlet type of proof which appears in §2 above, but not the improved new proof of §3. For the form G of §7, he treats only numbers not divisible by 5.

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*Note to appear in Proceedings of the National Academy.

†*Ueber die Darstellung ganzer Zahlen als Summen von sieben Kuben*, Dissertation, Göttingen, 1925.

A DIOPHANTINE AUTOMORPHISM*

BY E. T. BELL

1. *Introduction.* We define a *diophantine automorphism* of a form $f \equiv f(x, y, \cdots, w)$ to be a transformation of f into f^n, where $n>1$ is an integer, by means of a substitution of the type

$$x = X(x, y, \cdots, w), \qquad y = Y(x, y, \cdots, w), \cdots,$$
$$w = W(x, y, \cdots, w),$$

so that

$$f(X, Y, \cdots, W) = f^n(x, y, \cdots, w),$$

where $X, Y, \cdots, W$ are rational integral functions, with rational integral coefficients, of $x, y, \cdots, w$ and the coefficients of f.

Obviously powers of norms of algebraic integers, powers of sums of 4 or 8 squares, the generalizations of the latter which are composable forms and, generally, any power greater than the first of any composable form, have the automorphic property just defined. Excluding powers of composable forms we shall call any other form having a diophantine automorphism *proper*.

Examples of proper forms are extremely rare; their interest for diophantine analysis is evident. Apparently there are known but two instances, the discriminant of a binary cubic, this observation being due to Eisenstein† (see §9), and Cayley's‡ generalization of this to a certain octenary quartic with numerical coefficients $(1, -2, 4)$.

Being given a proper automorphism for a form with numerical coefficients, we devise a method of generalization

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* Presented to the Society, San Francisco Section, October 30, 1926.

† F. M. G. Eisenstein, Crelle's Journal, vol. 27 (1844), p. 105.

‡ A. Cayley, Collected Mathematical Papers, vol. 1, pp. 89–91; p. 113; p. 353; p. 532.

which yields a proper form with literal coefficients, whose automorphism degenerates to that of the given form when each of the literal coefficients is replaced by unity. The method leads to systems of linear diophantine equations and inequalities whose solution gives the required generalization. Both Eisenstein's and Cayley's automorphisms can be generalized in this way. The ordinary diophantine problem for Cayley's is so complicated, however, (leading as one detail to a system of 76 equations and inequalities in 8 variables), that we shall reproduce here only the work for Eisenstein's. The method extends and systematizes that used in a former paper* to generalize the 8 square identity, and it is applicable to all problems discussed there.

2. *Matric Exponents and Multipliers of a Transformation.* Let a, b, c be constant integers >0, and $u_j, v_j, r_j (j=1, \cdots, g)$ variable integers $\geqq 0$. Let $\phi_j (j=1, \cdots, g)$ be independent parameters. Write, symbolically,

$$\prod_{j=1}^{g} \phi_j^{au_j} \equiv \phi^{au},$$

and call au the *matric exponent* of ϕ. Let $\phi^{au}, \phi^{bv}, \phi^{cr}$ be any three such symbolic powers between which there is the relation

$$\phi^{au}\phi^{bv} = \phi^{cr},$$

the indicated multiplication on the left being as in common algebra, so that

$$au_j + bv_j = cr_j, \qquad (j = 1, \cdots, g).$$

Then we shall write these g equations as the single equation

$$au + bv = cr$$

between matric exponents, and similarly for matric equations in any number of variables $u, v, r, s, \cdots, x$.

The laws of combination of such equations are abstractly identical with those for ordinary linear systems, being merely a translation of the laws of algebraic exponents. It is unneces-

* E. T. Bell, Annals of Mathematics, (2), vol. 27 (1925), p. 99.

sary to elaborate them formally as they will be clear from the application made later.

Let $x, y, \cdots, z$ be independent variables, and $a, b, \cdots, c, u, v, \cdots, r, \phi$ as before. Then $\phi^{au}, \phi^{bv}, \cdots, \phi^{cr}$ will be called the *multipliers* of the transformation which replaces $x, y, \cdots, z$ by $\phi^{au}x, \phi^{bv}y, \cdots, \phi^{cr}z$ respectively.

In generalizing automorphisms we deal with a great many multipliers, matric exponents and variables simultaneously, and it is a saving of labor, also a suggestive impulse to the algebra to use a device which will show at a glance the specific variable to which a given exponent or multiplier appertains.

We shall write $\phi^x, \phi^y, \cdots, \phi^z$ for the multipliers appertaining to the variables $x, y, \cdots, z$. The exponents of these multipliers are $x, y, \cdots, z$. Finally, the multipliers will be denoted by their exponents. Thus $x, y, \cdots, z$ will denote either variables, the respective multipliers of these variables, or the matric exponents of the latter. No confusion can result from this triple interpretation of $x, y, \cdots, z$, as in each case it will be stated what the letters are. This supple device enables us to follow the multiplicative transformations of a set of variables without elaborate notations and with a minimum of algebra. It also reduces the particular kind of diophantine problem occurring in the subject from degree $n>1$ to degree 1.

3. *Eisenstein's Automorphism.* As we start from Eisenstein's identity we transcribe it here. Write

$$f(x,y,z,w) \equiv x^2w^2 - 3y^2z^2 + 4xz^3 + 4wy^3 - 6xyzw,$$

$$X \equiv 3xyz - x^2w - 2y^3$$

$$Y \equiv 2xz^2 - xyw - y^2z,$$

$$Z \equiv xzw - 2y^2w + yz^2,$$

$$W \equiv xw^2 - 3yzw + 2z^3,$$

so that

$$X = -\frac{1}{2}\frac{\partial f}{\partial w}, \quad Y = \frac{1}{6}\frac{\partial f}{\partial z}, \quad Z = -\frac{1}{6}\frac{\partial f}{\partial y}, \quad W = \frac{1}{2}\frac{\partial f}{\partial x}.$$

The automorphism is then

$$f(X,Y,Z,W) = f^2(x,y,z,w).$$

This striking result is today evident in many ways, for example by observing the discriminant of the cubicovariant of the binary cubic of which $f(x, y, z, w)$ is the discriminant. Nevertheless Eisenstein's remarks on it still stand, as no proof has indicated a method of attack on the general problem of finding all composable forms or upon the equally (or more) difficult one of diophantine automorphism which his theorem illustrates, and of which it was the first example. He says:

"Es scheint, man müsse dieser Gattung von identischen Gleichungen eine um so grössere Wichtigkeit zuschreiben, als man keine allgemeine Methode zu ihrer Auffindung besitzt, und weil dieselben namentlich in der Zahlenlehre oft die Grundlage zu ganzen Theorieen bilden; in welcher Hinsicht sich auf die Theorieen der quadratischen und ternären Formen, so wie auf diejenige der Zerlegung einer Zahl in die Summe von vier Quadraten und andere verweisen lasst."

It was proved by Dickson* that f is proper in the sense of §1.

4. *Summary of Extension.* We put here for convenience, in ordinary notation, the conclusion which follows from the subsequent algebra in §8. Let ϕ_j $(j=1, \cdots, g)$ be parameters, l_j, m_j, n_j, r_j arbitrary integers $\geqq 0$, such that

$$2m_j = l_j + n_j + r_j, \qquad (j = 1, \cdots, g),$$

and write

$$\lambda \equiv \prod_{j=1}^{g} \phi_j^{l_j}, \qquad \mu \equiv \prod_{j=1}^{g} \phi_j^{m_j}, \qquad \nu \equiv \prod_{j=1}^{g} \phi_j^{n_j},$$

$$\rho \equiv \prod_{j=1}^{g} \phi_j^{r_j}.$$

Then

$$[\mu^2(\lambda xw - yz)^2 - 4\lambda(\mu xz - \rho y^2)(\mu yw - \nu z^2)]^3$$
$$= \mu^2(\lambda XW - YZ)^2 - 4\lambda(\mu XZ - \rho Y^2)(\mu YW - \nu Z^2),$$

* L. E. Dickson, Comptes Rendus du Congrès International des Mathématiciens, Strassbourg, 1921, (pub. Toulouse): *Homogeneous polynomials with a multiplication theorem*, §9. The property of the Hessian noted by Haskell, American Mathematical Monthly, vol. 10 (1903) p. 2 and cited by Dickson, loc. cit., p. 9, was first proved by Cayley.

where

$$X \equiv 3\mu xyz - \lambda\mu x^2w - 2\rho y^3,$$
$$Y \equiv 2\lambda\nu xy^2 - \lambda\mu xyw - \mu y^2z,$$
$$Z \equiv \lambda\mu xzw - \lambda\rho y^2w + \mu yz^2,$$
$$W \equiv \lambda\mu xw^2 - 3\mu yzw + 2\nu z^3.$$

For $0=l_j=m_j=n_j=r_j$ $(j=1, \cdots, g)$ this becomes Eisenstein's identity in §3. An alternative form of the above is

$$\lambda^2[\nu\rho(\lambda xw - yz)^2 - 4(\mu xz - \rho y^2)(\mu yw - \nu z^2)]^3$$
$$= \nu\rho(\lambda XW - YZ)^2 - 4(\mu XZ - \rho Y^2)(\mu YW - \nu Z^2),$$

where λ, μ, ν, ρ are any parameters subject only to the condition

$$\mu^2 = \lambda\nu\rho.$$

For these parameters we have in fact

$$\mu^2(\lambda xw - yz^2) - 4\lambda(\mu xz - \rho y^2)(\mu yw - \nu z^2)$$
$$= \lambda[\nu\rho(\lambda xw - yz)^2 - 4(\mu xz - \rho y^2)(\mu yw - \nu z^2)],$$

and obviously the original λ, μ, ν, ρ (in terms of the ϕ_j) satisfy $\mu^2=\lambda\nu\rho$.

Although it is not of the type discussed in this paper, we may note in passing the following curious identity:

$$(px^2w^2 - 3qy^2z^2 + 4rxz^3 + 4swy^3 - 6txyzw)^3$$
$$= \frac{1}{p}X^2W^2 - \frac{3}{q}Y^2Z^2 + \frac{4}{r}XZ^3 + \frac{4}{s}WY^3 - \frac{6}{t}XYZW,$$

where

$$X \equiv 3txyz - px^2w - 2sy^3,$$
$$Y \equiv 2rxz^2 - txyz - qy^2z,$$
$$Z \equiv txzw - 2sy^2w + qyz^2,$$
$$W \equiv pxw^2 - 3tyzw + 2rz^3,$$

and p, q, r, s, t are any parameters such that

$$(pq - t^2)(rs - qt) = 0.$$

This is a consequence of either set of formulas in §§ 3, 4.

5. *Transformation of f.* Transform f in §3 by

$$\Sigma \equiv \begin{pmatrix} x, & y, & z, & w \\ x\phi^x, & y\phi^y, & z\phi^z, & w\phi^w \end{pmatrix},$$

the exponents of ϕ being matric, into

$$F \equiv Px^2w^2 - 3Qy^2z^2 + 4Rxz^3 + 4Swy^3 - 6Txyzw$$

and let the respective matric exponents of P, Q, R, S, T be p, q, r, s, t. Then

$$\begin{aligned} p &= 2x + 2w, \\ q &= 2y + 2z, \\ r &= x + 3z, \\ s &= w + 3y \\ t &= x + y + z + w, \end{aligned} \tag{1}$$

a set of diophantine matric equations.

Under Σ the matric exponents of the respective terms, in the order in which they occur, in X, Y, Z, W of §3 are

$$\begin{array}{lll} x + y + z\,, & 2x + w, & 3y, \\ x + 2z, & x + y + w, & 2y + z, \\ x + z + w\,, & 2y + w, & y + 2z, \\ x + 2w, & y + z + w\,, & 3z, \end{array} \tag{2}$$

and the multipliers having these respective exponents are

$$\begin{array}{lll} xyz\,, & x^2w\,, & y^3\,, \\ xz^2\,, & xyw\,, & y^2z\,, \\ xzw\,, & y^2w\,, & yz^2\,, \\ xw^2\,, & yzw\,, & z^3\,. \end{array} \tag{3}$$

6. *Conditions for Diophantine Automorphism.* As in the theory of numbers, if $a, b, c, \cdots$ are integral elements of any ray, and the quotient of b by a is an element of the ray, say c, we write $a \mid b$ for "a divides b." Hence $a \mid b$ implies that there exists an integral element c such that $b = ac$. Note that we are concerned throughout with arithmetic division and not merely algebraic. As in algebra a/b is the result of dividing a by b.

In order that F of §5 shall have a diophantine automorphism it is sufficient by (3) that the following 12 relations of divisibility for multipliers shall subsist,

$$(4)\qquad \begin{aligned} x &\mid (xyz,\ x^2w,\ y^3),\\ y &\mid (xz^2,\ xyw,\ y^2z),\\ z &\mid (xzw,\ y^2w,\ yz^2),\\ w &\mid (xw^2,\ yzw,\ z^3). \end{aligned}$$

Passing from (4) to matric exponents we see that the following must be integers $\geqq 0$:

$$(5)\qquad \begin{array}{lll} y+z, & x+w, & 3y-x,\\ x+2z-y, & x+w, & y+z,\\ x+w, & 2y+w-z, & y+z,\\ x+w, & y+z, & 3z-w. \end{array}$$

Hence it is sufficient that the following inequalities shall hold in integers (matric exponents) $\geqq 0$,

$$(6)\qquad 3y \geqq x, \quad x+2z \geqq y, \quad w+2y \geqq z, \quad 3z \geqq w.$$

The problem is therefore reduced to satisfying (1), (6) in integers $x, y, z, w, p, q, r, s, t \geqq 0$, the general solution being required.

7. *Solution of* (1), (6). From (1) we have

$$(7)\qquad p+q=2t, \qquad r+s=q+t,$$

and hence from (1)

$$(8)\qquad \begin{aligned} 6y &= 2x+2s-p,\\ 3z &= r-x,\\ 2w &= p-2x,\\ 3q &= 2r+2s-p,\\ 3t &= r+s+p, \end{aligned}$$

which must be solved in integers $\geqq 0$ subject to (6).

Treat x, r, p, s as parameters. Then, for integrality we must have

$$\begin{aligned} 2x + 2s &\equiv p, && \mod 6, \\ r &\equiv x, && \mod 3, \\ p &\equiv 0, && \mod 2, \\ 2r + 2s &\equiv p, && \mod 3, \\ r + s + p &\equiv 0, && \mod 3. \end{aligned} \tag{9}$$

From the first of (9) we get $p \equiv 0, \mod 2$; hence the third is redundant. Now if $p = 2k$, then $p \equiv 4p, \mod 6$, and hence from the first, $2x + 2s \equiv 4p, \mod 6$; thus, by the second, $r + s \equiv 2p, \mod 3$, and this implies the fifth. The set (9) is therefore equivalent to

$$2x + 2s \equiv p, \mod 6, \qquad r \equiv x, \mod 3; \tag{10}$$

and from (6), (8) we get

$$2r \geqq p, \quad 2s \geqq p, \quad 4r + p \geqq 2s, \quad 4s + p \geqq 2r \tag{11}$$

and we have also

$$q = (2r + 2s - p)/3, \qquad t = (r + s + p)/3. \tag{12}$$

The array (5) now becomes

$$\begin{array}{lll} (2r - p + 2s)/6, & p/2, & (-p + 2s)/2, \\ (4r + p - 2s)/6, & p/2, & (2r - p + 2s)/6, \\ p/2, & (-2r + p + 4s)/6, & (2r - p + 2s)/6, \\ p/2, & (2r - p + 2s)/6, & (-p + 2r)/2. \end{array} \tag{13}$$

If (10), (11) are satisfied, it follows from the way in which (13) was obtained that its 12 members are integers $\geqq 0$. To exhibit this solution in terms of ordinary algebra we make some reductions.

8. *Transition to Ordinary Algebra.* By §5, beginning, we have

$$(P, Q, R, S, T) = (\phi^p, \phi^q, \phi^r, \phi^s, \phi^t);$$

hence the equivalent in terms of multipliers of (13) is

$$(14)\qquad \begin{matrix} (R^2S^2/P)^{1/6}, & P^{1/2}, & (S^2/P)^{1/2}, \\ (PR^4/S^2)^{1/6}, & P^{1/2}, & (R^2S^2/P)^{1/6}, \\ P^{1/2}, & (PS^4/R^2)^{1/6}, & (R^2S^2/P)^{1/6}, \\ P^{1/2}, & (R^2S^2/P)^{1/6}, & (R^2/P)^{1/2}, \end{matrix}$$

and from (12) we have

$$(15)\qquad Q = (R^2S^2/P)^{1/3}, \qquad T = (PRS)^{1/3},$$

and P, R, S are not similarly reducible, since we took x, p, r, s as independent (matric) parameters.

As a check, we should have, by (7),

$$(16)\qquad (PQ - T^2)(RS - QT) = 0,$$

a relation used in what follows, and this is verified by (15).

Applying (16) to (14), we get the array in the form

$$(17)\qquad \begin{matrix} Q^{1/2}, & P^{1/2}, & S/P^{1/2}, \\ R/Q^{1/2}, & P^{1/2}, & Q^{1/2}, \\ P^{1/2}, & S/Q^{1/2}, & Q^{1/2}, \\ P^{1/2}, & Q^{1/2}, & R/P^{1/2}, \end{matrix}$$

whose twelve members must be rational integral functions of parameters together with Q, T as given by (15).

For rationality and integrality (17) demands

$$(18)\qquad Q = Q_1^2, \quad P = P_1^2, \quad S = S_1P_1 = S_2Q_1, \quad R = R_1Q_1 = R_2P_1,$$

the letters with suffixes being new parameters. From (18) we get

$$(19)\qquad Q_1^6\,P_1^2 = R^2S^2, \qquad T^3 = RSP_1^3, \qquad P_1/Q_1 = S_2/S_1 = R_1/R_2 = K,$$

where K is a new parameter. Hence

$$(20)\qquad P = K^2Q_1^3, \quad Q = Q_1^2, \quad R = KQ_1R_2, \quad S = KQ_1S_1, \quad T = KQ_1^2,$$

expressing P, Q, R, S, T in terms of 4 parameters K, Q_1, R_2, S_1 between which is the single relation

$$Q_1^2 = KR_2S_1. \tag{21}$$

Finally then, making the change in notation

$$\lambda \equiv K, \qquad \mu \equiv Q_1, \qquad \nu \equiv R_2, \qquad \rho \equiv S_1,$$

we have the results summarized in §§ 3, 4.

The use of the symbolic method is a rapid means for proving the existence or non-existence of a diophantine automorphism; the passage to ordinary algebra is then a simple translation of the matric and multiplier equations of the symbolic solution. Notice that the symbolic method avoids all irrationalities; its equations and all of its processes are carried out in terms of positive integers.

9. *References.* Many of Cayley's papers contain references to Eisenstein's theorem and to his own generalization of it, including the geometry of the related developable surface in homogeneous coordinates. Cayley's form is of interest from another point of view; it is that (an octenary quartic) which occurs incidentally in section 235 of the *Disquisitiones Arithmeticae* in connection with the composition of binary quadratic forms.

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THE CAUCHY-HEAVISIDE EXPANSION FORMULA AND THE BOLTZMANN-HOPKINSON PRINCIPLE OF SUPERPOSITION*

BY F. D. MURNAGHAN

1. *Introduction.* It is the object of the present note to point out that the well known expansion theorem of Heaviside is an immediate corollary of the classical method given by Cauchy for finding a particular solution of a linear, non-homogeneous, ordinary differential equation when the general solution of the corresponding homogeneous equation is known. Various proofs of Heaviside's theorem have been given by Bromwich and Wagner, using contour integrals, by Carson and others and it seems hardly possible that the direct connection with Cauchy's method can have escaped attention. This method is interesting, furthermore, since it furnishes a rigorous proof of the useful Boltzmann-Hopkinson principle of superposition.† The explicit form of the expansion theorem in the case when the characteristic equation has multiple roots is also given.

2. *Cauchy's Method.* Although Cauchy's method is classical and to be found in the older books, it does not seem to find a place in the more modern texts and it will, therefore, be convenient to give an indication of its derivation. Let us suppose

* Presented to the Society, under a different title, September 9, 1926.

† The following citations may serve to give a proper orientation: Bromwich, T. J., *Normal coordinates in dynamical systems*, Proceedings of the London Society, vol. 15 (1916), pp. 401–448. Wagner, K. W., *Über eine Formel von Heaviside zur Berechnung von Einschaltvorgängen*, Archiv für Elektrotechnik, vol. 4 (1916), pp. 159–193. Carson, J. R., *On a general expansion theorem for the transient oscillations of a connected system*, Physical Review, vol. 10 (1917), pp. 217–225. Carson, J. R., *Theory of the transient oscillations of electrical networks and transmission lines*, Transactions A. I. E. E., vol. 38 (1919), pp. 345–427.

that we wish to find a particular solution of the linear differential equation of the nth order

$$(1) \qquad \phi(D)u = f(t) \; ; \qquad \phi(D) \equiv a_0 D^n + \cdots + a_n,$$

where u is the dependent variable, t the independent variable and $D \equiv d/dt$. The coefficients a are either constants or functions of t. If $(v_1, \cdots, v_n)$ are distinct solutions of the corresponding homogeneous equation

$$(1') \qquad \phi(D)v = 0,$$

its general solution is

$$(2) \qquad v = C_1 v_1 + \cdots + C_n v_n,$$

where the C's are arbitrary constants. We choose these so that v and its derivatives with respect to t up to the $(n-2)$d vanish, when $t=\tau$ the $(n-1)$st derivative taking the value $f(\tau)/a_0(\tau)$; τ being a fixed but arbitrary value of t. To find the proper values for the C's, we have merely to solve a system of n linear equations whose determinant is the Wronskian $\left|v_1 v_2' \cdots v_n^{(n-1)}\right|$ and this does not vanish on account of the distinctness of the v's. When the values of the C's as thus determined are entered in (2), v becomes a function of t and τ, and to stress this we may write it as $v(t, \tau)$. Then Cauchy's contention is that the integral

$$(3) \qquad u(t) = \int_0^t v(t,\tau)d\tau$$

is a particular solution of (1). We have, in fact, from (3)

$$Du = \int_0^t Dv(t,\tau)d\tau + v(t,t) = \int_0^t Dv(t,\tau)d\tau,$$

since $v(\tau, \tau)$ vanishes by definition for every value of τ. Similarly

$$D^2u = \int_0^t D^2v(t,\tau)d\tau, \cdots, D^{n-1}u = \int_0^t D^{n-1}v(t,\tau)d\tau,$$

while

$$D^n u = \int_0^t D^n v(t,\tau)d\tau + \left\{D^{n-1}v(t,\tau)\right\}_{\tau=t} = \int_0^t D^n v(t,\tau)d\tau + \frac{f(t)}{a_0(t)},$$

so that

$$\phi(D)u = \int_0^t \phi(D)v(t,\tau)d\tau + f(t) = f(t).$$

The particular solution u of (1) obtained in this way is characterized by the fact that it vanishes with its derivatives up to the $(n-1)$st when $t=0$; the nth derivative having, consequently, the value $f(0)/a_0(0)$ when $t=0$. It is apparent from the definition of the C's occurring in (2) that they may be obtained from the values they would have if $f(\tau)$ were $\equiv 1$ by multiplication by $f(\tau)$. Hence if we denote by $\psi(t,\tau)$ the particular function $v(t,\tau)$ corresponding to the particular problem where $f(t)\equiv 1$ we have $v(t,\tau) \equiv f(t)\psi(t,\tau)$, and (3) may be written in the form

$$(3^{\text{bis}}) \qquad u(t) = \int_0^t f(\tau)\psi(t,\tau)d\tau.$$

Denoting by $\xi(t,0)$ the particular solution corresponding to $f(t)\equiv 1$ we have

$$\xi(t,0) \equiv \int_0^t \psi(t,\tau)d\tau,$$

which suggests the notation

$$(4) \qquad \xi(t,\tau) \equiv \int_\tau^t \psi(t,s)ds,$$

with the relation

$$\frac{\partial \xi(t,\tau)}{\partial \tau} = -\psi(t,\tau).$$

An integration of (3^{bis}) by parts then yields

$$(5) \qquad \begin{cases} u(t) = [-f(t)\xi(t,t) + f(0)\xi(t,0)] + \displaystyle\int_0^t \xi(t,\tau)\frac{df}{d\tau}d\tau \\ \qquad = f(0)\xi(t,0) + \displaystyle\int_0^t \xi(t,\tau)\frac{df}{d\tau}d\tau, \quad \text{since} \quad \xi(t,t) = 0. \end{cases}$$

This is the mathematical statement of the Boltzmann-Hopkinson principle of superposition according to which we are able

to build up from a particular solution $\xi(t, \tau)$ of the differential equation (1) when $f(t) \equiv 1$, a particular solution of this equation when $f(t)$ is an arbitrary function of t. The particular solution obtained in this way is characterized by the fact that it vanishes, together with its derivatives with respect to t up to the $(n-1)$st, when $t=0$.

3. *The Particular Case when the Coefficients of $\phi(D)$ are Constant.* In the first case we shall consider the general case where the zeros of the polynomial $\phi(D)$ are all distinct and we shall write

$$\phi(D) \equiv a_0(D - r_1)(D - r_2) \cdots (D - r_n).$$

We may choose as our n distinct solutions of (1′) the functions

$$v_1 = e^{r_1(t-\tau)}, \cdots, v_n = e^{r_n(t-\tau)},$$

and our equations to determine the constants C of (2) are

$$(6) \qquad \begin{cases} C_1 + \cdots + C_n = 0, \\ r_1C_1 + \cdots + r_nC_n = 0, \\ \cdots\cdots\cdots\cdots\cdots \\ r_1^{n-1}C_1 + \cdots + r_n^{n-1}C_n = \dfrac{f(\tau)}{a_0}. \end{cases}$$

These equations show that $(C_1, \cdots, C_n)$ are the coefficients of the analysis of $f(\tau)/\phi(D)$ into its simple fractions. In fact if we write

$$\frac{f(\tau)}{\phi(D)} \equiv \frac{C_1}{D - r_1} + \cdots + \frac{C_n}{D - r_n},$$

and expand each of the fractions on the right-hand side near $D=\infty$, we find on observing that $D=\infty$ is a zero of order n of the left-hand side, the first term in its expansion being $f(\tau)/(a_0D^n)$, exactly the equations (6) to determine the C's. Further more on observing that $D=r_p$ is a simple pole with residue $f(\tau)/\phi'(r_p)$ of the left-hand side we obtain the relations

$$(7) \qquad C_p = \frac{f(\tau)}{\phi'(r_p)}, \qquad (p = 1, \cdots, n).$$

Hence the function $v(t, \tau)$ is

$$v(t,\tau) \equiv f(\tau) \sum_{p=1}^{n} \frac{e^{r_p(t-\tau)}}{\phi'(r_p)},$$

so that

$$\psi(t,\tau) \equiv \sum_{p=1}^{n} \frac{e^{r_p(t-\tau)}}{\phi'(r_p)},$$

and

$$\xi(t,\tau) \equiv \sum_{p=1}^{n} \frac{e^{r_p(t-\tau)}}{r_p\phi'(r_p)} - \sum_{1}^{n} \frac{1}{r_p\phi'(r_p)}.$$

The relation

$$\frac{1}{\phi(D)} \equiv \sum_{p=1}^{n} \frac{1}{\phi'(r_p)(D - r_p)}$$

obtained above shows that

$$\frac{1}{\phi(0)} = -\sum_{p=1}^{n} \frac{1}{r_p\phi'(r_p)},$$

so that we may write

$$(8) \qquad \xi(t,\tau) = \frac{1}{\phi(0)} + \sum_{p=1}^{n} \frac{e^{r_p(t-\tau)}}{r_p\phi'(r_p)},$$

and, in particular,

$$(9) \qquad \xi(t) \equiv \xi(t,0) = \frac{1}{\phi(0)} + \sum_{p=1}^{n} \frac{e^{r_p t}}{r_p\phi'(r_p)}.$$

It is this particular form of Cauchy's result which yields immediately the Heaviside expansion theorem. Before proceeding to show this it will be well, however, to see what modification is introduced when two or more of the zeros of $\phi(D)$ coincide. For simplicity of presentation let us suppose that $r_2 = r_1$, the other roots being all different from r_1 and from one another. Then we take

$$v_1 = e^{r_1(t-\tau)}, \quad v_2 = (t-\tau)e^{r_1(t-\tau)}, \quad v_3 = e^{r_3(t-\tau)}, \quad \cdots, \quad v_n = e^{r_n(t-\tau)},$$

and the equations to determine the constants C are

$$\begin{aligned}
C_1 + 0 \cdot C_2 + C_3 + \cdots + C_n &= 0, \\
r_1C_1 + 1C_2 + r_3C_3 + \cdots + r_nC_n &= 0, \\
r_1^2C_1 + 2r_1C_2 + r_3^2C_3 + \cdots + r_n^2C_n &= 0, \\
\cdots\cdots\cdots

the coefficient of C_2 in each equation being the derivative with respect to r_1 of the coefficient of C_1. These equations show, just as before, that $(C_1, \cdots, C_n)$ are the coefficients in the analysis of $f(\tau)/\phi(D)$ into its simple fractions; i.e.,

$$\frac{f(\tau)}{\phi(D)} \equiv \frac{C_1}{D-r_1} + \frac{C_2}{(D-r_1)^2} + \frac{C_3}{D-r_3} + \cdots + \frac{C_n}{D-r_n}.$$

Developing this identity in the neighborhood of the double zero $D=r_1$, and setting

$$\frac{(D-r_1)^2}{\phi(D)} \equiv F(D),$$

we have

$$C_1 = f(\tau)F'(r_1), \qquad C_2 = f(\tau)F(r_1).$$

The part of the function $\xi(t)$ corresponding to the double zero $r=r_1$ may be conveniently expressed as follows. Writing $C_p=f(\tau)A_p$, $(p=1, \cdots, n)$, so that

$$(10) \qquad \frac{1}{\phi(D)} = \frac{A_1}{D-r_1} + \frac{A_2}{(D-r_1)^2} + \cdots + \frac{A_n}{D-r_n},$$

we have, for the part of $\xi(t)$ in question, to evaluate the integral

$$\int_0^t \{A_1 + A_2(t-\tau)\} e^{r_1(t-\tau)} d\tau.$$

If we write $t-\tau=s$, this becomes

$$\int_0^t (A_1 + A_2 s) e^{r_1 s} ds = \frac{A_2}{r_1} t e^{r_1 t} + \left(\frac{A_1}{r_1} - \frac{A_2}{r_1^2}\right) e^{r_1 t} - \left(\frac{A_1}{r_1} - \frac{A_2}{r_1^2}\right);$$

as $A_1=F'(r_1)$ and $A_2=F(r_1)$, the variable part of this may be written in the form

$$\frac{F(r_1)}{r_1} t e^{r_1 t} + \left\{ \frac{d}{dr_1}\left(\frac{F(r_1)}{r_1}\right)\right\} e^{r_1 t}.$$

In an exactly similar manner it can be shown that the variable part of $\xi(t)$ corresponding to a multiple root of order m is

$$\frac{1}{(m-1)!}\left[\left(\frac{F(r_1)}{r_1}\right)t^{m-1}e^{r_1 t} + (m-1)\left\{\frac{d}{dr_1}\left(\frac{F(r_1)}{r_1}\right)\right\}t^{m-2}e^{r_1 t} + \cdots + \left\{\frac{d^{m-1}}{dr_1{}^{m-1}}\left(\frac{F(r_1)}{r_1}\right)\right\}e^{r_1 t}\right],$$

where $F(D) \equiv (D-r_1)^m/\phi(D)$; the expression in parenthesis being written with binomial coefficients. Thus in the case of a triple root the constants C are given by the equation

$$\frac{f(\tau)}{\phi(D)} \equiv \frac{C_1}{D-r_1} + \frac{C_2}{(D-r_1)^2} + \frac{2!C_3}{(D-r_1)^3} + \cdots.$$

It follows from (10), on putting $D=0$, that the constant part of $\xi(t)$ is $1/\phi(0)$. These results furnish the general expression for the particular solution of $\phi(D)u = 1$ which vanishes with its derivatives up to the $(n-1)$st when $t=0$.

4. *The Heaviside Problem.* In his study of electric oscillations Heaviside had to consider a system of ordinary linear differential equations of the type

$$\sum_{\alpha=1}^{k} a_{r\alpha}x_\alpha = E_r(t), \qquad (r = 1, \cdots, k). \tag{11}$$

Here the a's are polynomials in the operator D whose coefficients may involve the independent variable t (although Heaviside was particularly interested in the case where these coefficients are constants) and expressions for the k dependent variables are sought for, the functions $E(t)$ on the right-hand sides of the equations (11) being given. All the dependent variables but one, x_s say, may be eliminated by operating on the equations (11) with the cofactors of the sth column of the determinant $|a_{r\alpha}| \equiv \phi(D)$. Denoting these cofactors by A_{1s}, $\cdots$, A_{ks} we find

$$\phi(D)x_s = \sum_{r=1}^{k} A_{rs}E_r(t).$$

Let us now confine our attention to the case where $E_p(t)=1$, all other E's being zero, and where, in addition, the coefficients

of the polynomials a_{rs} are constants. In this event, the operators $\phi(D)$ and A_{ps} are commutative and we first set up that particular solution of

$$\phi(D)y_p = 1$$

which vanishes, together with its derivatives up to that of the $(n-1)$st order, when $t=0$, n being the degree of the polynomial $\phi(D)$. Then $x_{sp}=A_{ps}y_p$ is a particular solution of the equation $\phi(D)x_s=A_{ps}$ and it is characterized by the fact that its derivatives up to an order one less than the degree in D of the elements a_{rs} vanish when $t=0$. From the exponential form already given for $y_p(t,0)$ in the case where the zeros of $\phi(D)$ are simple, it follows that

$$(12) \qquad x_{sp}(t,0)=A_{ps}y_p(t,0) = \sum_{\alpha=1}^{n} \frac{A_{ps}(r_\alpha)e^{r_\alpha(t)}}{r_\alpha\phi'(r_\alpha)} + \frac{A_{ps}(0)}{\phi(0)},$$

which is Heaviside's expansion formula. If now we consider the equation $\phi(D)u_p = E_p(t)$, a particular solution is

$$u_p(t) = E_p(0)y_p(t,0) + \int_0^t \frac{dE_p}{d\tau} y_\tau(t,\tau)d\tau,$$

this particular solution being characterized by the fact that it, together with all its derivatives up to the $(n-1)$st, vanish when $t=0$. The function

$$\bar{x}_{sp} = A_{ps}u_p(t) = E_p(0)x_{sp}(t,0) + \int_0^t \frac{dE_p}{d\tau}\, x_{sp}(t,\tau)d\tau$$

is, accordingly, a particular solution of the equation

$$\phi(D)\bar{x}_{sp} = A_{ps}E_p(t)\,.$$

Adding with respect to p from 1 to k, we find that the set of functions

$$(13) \qquad x_s = \sum_{p=1}^{k} \left\{ E_p(0)x_{sp}(t,0) + \int_0^t \frac{dE_p}{d\tau} x_{sp}(t,\tau)d\tau \right\}$$

furnish that particular solution of the equations(11) which is

characterized by the fact that all the x's and their derivatives up to an order one less than the degree of the a_{rs} in D vanish when $t=0$.

When the zeros of $\phi(D)$ are not distinct we avail ourselves of the relations

$$D(te^{rt}) = rte^{rt} + e^{rt}, \quad D(t^m e^{rt}) = rt^m e^{rt} + mt^{m-1}e^{rt},$$

$$D^2(t^m e^{rt}) = r^2 t^m e^{rt} + 2rmt^{m-1}e^{rt} + m(m-1)t^{m-2}e^{rt},$$

$$D^3(t^m e^{rt}) = r^3 t^m e^{rt} + 3r^2 mt^{m-1}e^{rt} + 3rm(m-1)t^{m-2}e^{rt} + m(m-1)(m-2)t^{m-3}e^{rt},$$

which show that

$$A_{ps}(t^m e^{rt}) = A_{ps}(r)\, t^m e^{rt} + A'_{ps}(r)mt^{m-1}e^{rt} + A''_{ps}(r)\frac{m(m-1)}{1\cdot 2}t^{m-2}e^{rt} + \cdots.$$

When, then, we are operating with A_{ps} on a term such as

$$\frac{1}{(m-1)!}\left\{\frac{F(r_1)}{r_1}t^{m-1}e^{r_1 t} + (m-1)\frac{d}{dr_1}\frac{F(r_1)}{r_1}t^{m-2}e^{r_1 t} + \cdots\right\}$$

which corresponds to a multiple root r_1 of order m, we obtain

$$\frac{1}{(m-1)!}\left\{\frac{A_{ps}(r_1)F(r_1)}{r_1}t^{m-1} + (m-1)\frac{d}{dr_1}\left(\frac{A_{ps}(r_1)F(r_1)}{r_1}\right)t^{m-2} + \frac{(m-1)(m-2)}{1\cdot 2}\frac{d^2}{dr_1^2}\left(\frac{A_{ps}(r_1)F(r_1)}{r_1}\right)t^{m-3} + \cdots\right\}e^{r_1 t};$$

and these terms must be used in the expansion formula (12).

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SINGULARITIES OF THE HESSIAN*

BY T. R. HOLLCROFT

1. *Introduction.* It has been proved† that when a curve f has no point singularities its Hessian H has no point singularities. Then point singularities occur on H when and only when f has point singularities. Moreover, singular points of H can occur only at those points which are singular points of f.

The number of intersections of f and H at any singularity of f is

$$6\delta_1 + 8\kappa_1 + \iota_1$$

where δ_1, κ_1, ι_1 are the numbers of nodes, cusps, and inflections respectively contained in the singularity of f. A given singularity of f needs but to be resolved and the number of intersections of f and H are thus found without reference to H. In order for this number of intersections to occur, there must be a singularity of H at this point, but except for cusps and simple multiple points with distinct tangents these singularities of H have not been investigated.

The purpose of this paper is to explain geometrically how the intersections of f and H at a given singularity of f occur. The principal problem involved is to find the singularity of H corresponding to a given singularity of f.

2. *Simple Multiple Points.* It has long been known that at a simple r-fold point of f with r distinct tangents, H has a $(3r-4)$-fold point, r of whose tangents coincide, one each, with r tangents of the r-fold point on f; also that at a cusp of f, H has a triple point two of whose tangents coincide with the cuspidal tangent.

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* Presented to the Society, October 30, 1926.

† A. B. Basset, *On the Hessian, the Steinerian, and the Cayleyan*, Quarterly Journal, vol. 47 (1916), p. 227.

Any number of cusps up to and including $r-1$ may occur in a simple r-fold point. The occurrence of κ_1 cusps in such a multiple point causes κ_1+1 tangents at that point to become consecutive. Also each cusp implies two additional intersections of f and H at this point.

Consider that f has an r-fold point at the origin O, the tangents at which are given by the equation $u_r=0$. Of the $3r-4$ tangents to H at O, r have the equation $u_r=0$ and the remaining $2(r-2)$ have the equation

$$T \equiv \frac{\partial^2 u_r}{\partial x^2} \cdot \frac{\partial^2 u_r}{\partial y^2} - \left(\frac{\partial^2 u_r}{\partial x\, \partial y}\right)^2 = 0.*$$

If the singularity of f at O contains $c-1$ cusps, it has c consecutive tangents. The tangents to f at O are now given by

$$u_r \equiv (ax + by)^c u_{r-c} = 0.$$

For this form of u_r, the equation T becomes

$$T \equiv (ax + by)^{2(c-1)} U_{2(r-c-1)} = 0,$$

where $U_{2(r-c-1)}$ is a function of $ax+by$, u_{r-c} and their first and second partial derivatives. Then each cusp in the r-fold point O of f causes two of the $2(r-2)$ additional tangents to H at O to become consecutive with the two consecutive tangents common to H and f.

If, in the above, $c=r$, the expression for T is identically satisfied so that it no longer is the equation of the additional tangents to H at O. This shows that the above conclusions are justified when and only when the number of cusps in the r-fold point of f does not exceed $r-2$. When it contains $r-2$ cusps, there are but two distinct tangents to f at P and these two are the only tangents to H at P. The $(3r-4)$-fold point on H now contains $3(r-2)$ cusps. If a general $(3r-4)$-fold point, it could contain one more cusp, causing all its tangents to become consecutive, but this special Hessian multiple point can not have this additional cusp, that is, a simple

* Hilton, *Plane Algebraic Curves*, 1920, Ex. 11, p. 103.

multiple point on a Hessian can not consist of a single superlinear branch.

The r-fold point P on f, however, may contain $r-1$ cusps and consist of a single superlinear branch. At P, H now has a $3(r-1)$-fold point with $2(r-1)$ of its tangents consecutive with the single tangent to f at P and the remaining $r-1$ tangents distinct from this and from each other. The well known triple point of H at a cusp of f is a special case of this for $r=2$.

Consider the r-fold point of f first without cusps and let nodes be changed into cusps one by one up to and including the maximum. Each cusp up to and including the $(r-2)$th causes three nodes of the $(3r-4)$-fold point of H to be converted into three cusps. The genus of H is not affected by this exchange so long as the number of cusps of f at P does not exceed $r-2$. When the $(r-1)$th node is changed into a cusp, nothing different from usual happens on f, but something very different occurs on H. The multiplicity of P on H is increased by unity causing the addition of $3r-4$ nodes and a corresponding decrease of $3r-4$ in the genus of H and, in addition, $r-2$ cusps in the multiple point of H at P are changed back into nodes causing $r-2$ branches of H at P that had been coincident to become distinct.

If $r=n-1$, H is of order $3(n-2)$ with a $(3n-7)$-fold point at P. In this case and in this case only can the Hessian be rational. If $r=n-1$, H is a proper curve when and only when the number of cusps of f at P does not exceed $n-3$. If the $(n-1)$-fold point of f contains $n-2$ cusps, H degenerates into $3(n-2)$ lines through P of which $2(n-2)$ coincide with the single tangent to f at P. A well known special case of this is the Hessian of a cuspidal cubic.

3. *Compound Singularities.* When the singularity of f at P is compound, the singularity of H at P is also compound.

Any number s of consecutive r-fold points at P on f with r distinct branches determines at P, on H, s consecutive $3(r-1)$-fold points with $3(r-1)$ distinct branches and with the same tangent as that to f at P. There are, then, at P, r branches of

f, each of which has contact of order s with each of the $3(r-1)$ branches of H that pass through P.

Any series of consecutive multiple points of orders $r_1, r_2, r_3, \cdots$ where $r_1 > r_2 \geqq r_3 \geqq \cdots$ and $r_1 > r_2 + 1$ gives rise to a series of consecutive multiple points on H of the respective orders $3r_1-4$, $3(r_2-1)$, $3(r_3-1), \cdots$ and through the same respective penultimate points and therefore with the same tangent. If, however, $r_2 \leqq r_1 \leqq r_2+1$, the series on H are of the respective orders $3(r_1-1)$, $3(r_2-1)$, $3(r_3-1), \cdots$.

In case $r_1 = r_2$, the $3(r_1-1)$ branches of H all have the same tangent at P as the r_1 branches of f. In case $r_1 = r_2+1$, at P on f there are r_2 branches with the same tangent and one branch with a distinct tangent. At P on H there are $3r_1-4$ branches that have the tangent common to the r_2 branches of f and the remaining branch of H has the same tangent as the remaining branch of f.

Since any series of consecutive multiple points of f gives rise to a series of consecutive multiple points of H through the same penultimate points respectively, the order of nearness* of any two consecutive points of H is always equal to the order of nearness of the two corresponding consecutive multiple points of f.

Any series of consecutive multiple points of orders $r_1, r_2, r_3, \cdots$ where $r_1 \geqq r_2 \geqq r_3 \geqq \cdots$ may contain any number of cusps up to and including r_1-1. For each node changed into a cusp in the part of the r_1-fold point involved in the consecution or in any of the other consecutive points, a bitangent of the singularity is changed into a stationary tangent. Then if the number of cusps k is such that

$$k \leqq r_2 - 1$$

the singularity contains k inflections, but if

$$r_1 - 1 \geqq k \geqq r_2 - 1$$

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* See F. Enriques, *Lezioni sulla Teoria Geometrica delle Equazioni e delle Funzioni Algebriche*, vol. 2, pp. 404–408.

the singularity contains r_2-1 inflections provided that the cusps are considered as being introduced so far as possible into that part of the singularity involved in the consecution. Then the introduction of $k \leqq r_2-1$ cusps causes the addition of $3k$ intersections of f and H at this point, $2k$ for the cusps that replace k nodes and k for the k inflections that replace k bitangents. The introduction of k cusps into the portion of the r_1-fold point not involved in the consecution causes the addition of only $2k$ intersections as for simple multiple points.

If the consecutive multiple points of f at P contain any number k of cusps all of which are involved in the consecution, the consecutive multiple points of H at P contain $3k-1$ cusps. All the branches of both f and H at P (except the simple branches of the r_1-fold point of f and the $(3r_1-4)$-fold point of H in case $r_1>r_2$) have the same tangent whether the points contain cusps or not. The change of k nodes into k cusps on f causes $k+1$ of the branches to coincide, while on H the $3k-1$ cusps cause the coincidence of $3k$ branches. H and f must now have $3k$ additional intersections at P. None of these are accounted for by the mere coincidence of the branches, but all are accounted for by the fact that the superlinear branches of f and H have closer contact than had each of the component branches when distinct. If there are s consecutive r-fold points of f at P, each branch of f has s-point contact with each branch of H at P. When the multiple points of f contain k cusps, however, each of the $k+1$ partial branches of the superlinear branch of f at P has $[(s+1)/(k+1)]$-point contact with each of the $3k$ partial branches of the superlinear branch of H at P.

A flecnode on f gives rise to a flecnode on H with the same stationary and simple tangents respectively. Also a biflecnode on f gives rise to a biflecnode on H with the same two stationary tangents. In general, if any of the tangents at an r-fold point of f are stationary tangents, these are also stationary tangents to H at this point.

4. *Line Singularities.* Simple line singularities of f cause on H either no singularities or simple line singularities. If the

line singularity of f be composed of only bitangents, that is, if all its contacts are distinct, H avoids both this line and its contacts and has no singularity caused by such a multiple line of f. Moreover, a multiple line of class ρ of f may contain any number up to $(\rho+1)/2$ inflections, no more than one at each contact except that one point of contact may contain two inflections, and H will have no line singularity. It is only when more than two inflections occur at one contact or more than one inflection at two or more contacts of a multiple line p of f that H has p as a multiple line.

In general, if f has a multiple line p of class ρ and q is the number of contacts of p with f that contain two or more inflections and i_j is the number of inflections contained in any one of these q contacts, then p is a multiple tangent of H of class

$$\rho_1 \equiv \sum_{j=1}^{q} i_j - q.$$

Since the number of inflections of a multiple line of class ρ with q distinct contacts can not exceed $\rho-q$, there is an upper limit to the class ρ_1 of the line p of H

$$\rho_1 \leqq \rho - 2q.$$

The equality sign holds only when no contacts of p with f contain less than two inflections. It is also apparent that the fewer the number of distinct contacts, the larger the class of the multiple line of H may be. Thus the maximum class of a multiple line of H that results from a multiple line of class ρ of f is $\rho-2$. This occurs when and only when p contains $\rho-1$ stationary tangents to f, in which case $q=1$. In this case, the multiple line p contains $\rho-3$ stationary tangents to H and all its contacts are at one point.

5. *Singularities that a Hessian Cannot Have.* In the preceding sections, the multiple points and lines that a Hessian can have were discussed. There are, however, certain singular points that a Hessian can not have under any circumstances. These include all simple or compound double points except

distinct nodes, multiple points of order $3a+1$ for a any integer and simple multiple points with all tangents consecutive.

There are no such restrictions on multiple lines of the Hessian.

6. *The Hessian as a Jacobian.* Since the Hessian of f is the Jacobian of the net of first polars of f, the singularity P of H caused by a given singularity P of f is the singularity of the Jacobian of the net of first polars of f corresponding to this basis point P on the first polars. Since an r-fold point P of f causes a $(3r-4)$-fold point P of H and an $(r-1)$-fold point P on each first polar, then H as the Jacobian has a $(3i-1)$-fold point at an i-fold basis point of the net. Since this result is general, it might be inferred that the singularities of the Jacobian corresponding to any kind of basis point could be now found by finding the kind of basis point on the first polars determined by a given singularity of f. However, the result is general only for simple multiple points with distinct branches. If the curves of the net have contact of any given order with each other at P, the nature of the singularity P on the Jacobian can not be predicted unless more is known about the net. For example, simple contact in a first polar net may result from a cusp on f or a tacnode on f. In these cases, the singularity on the Jacobian is respectively a triple point two of whose tangents coincide with the cuspidal tangent or two consecutive triple points whose tangent coincides with the tacnodal tangent. But a simple contact in a net of otherwise general curves causes on the Jacobian a triple point with three distinct tangents one of which coincides with the common tangent of the curves of the net.

For $n>3$, first polar nets are not the most general nets of order $n-1$ and for all values of n, if f has singularities, first polar nets are very highly specialized. For this reason (except in the case of a simple multiple point with distinct tangents) general results for the singularity of the Jacobian at a given singularity of the net can not be obtained by regarding the Hessian as the Jacobian of the net of first polars.

WELLS COLLEGE

THREE THEOREMS ON CLOSURE OF BIORTHOGONAL SYSTEMS OF FUNCTIONS*

BY R. E. LANGER

1. *Introduction.* Professor Birkhoff has given a theorem† which asserts the closure of a given normal orthogonal set of functions provided it is sufficiently near to another such set which is closed. In its applications this theorem is frequently used in conjunction with asymptotic forms for the functions involved, and the information at hand applies only to a portion of the entire set. For this reason an extension of the theorem to sets which involve a certain lack of closure is useful. Such an extension together with a generalization of the theorem to biorthogonal systems of functions is given below in Theorem I.

In Theorems II and III the discussion is restricted to the closure of such biorthogonal systems as are composed of the characteristic functions of an integral equation.

A set of continuous functions $\{u_n(x)\}$ will be said to involve a k-fold lack of closure on a given interval if there exist precisely k continuous functions not identically zero which are linearly independent and are orthogonal on the interval to every function of the set.

THEOREM I. *Let* $\{u_n(x), v_n(x)\}$, *that is,*

$$(1) \qquad \begin{cases} u_0(x), & u_1(x), & u_2(x), & \cdots, \\ v_0(x), & v_1(x), & v_2(x), & \cdots, \end{cases}$$

be a normalized biorthogonal system of continuous functio on the interval $a \leqq x \leqq b$, *and let* $\{\bar{u}_n(x), \bar{v}_n(x)\}$, *i. e.,*

$$(2) \qquad \begin{cases} \bar{u}_0(x), & \bar{u}_1(x), & \bar{u}_2(x), & \cdots, \\ \bar{v}_0(x), & \bar{v}_1(x), & \bar{v}_2(x), & \cdots, \end{cases}$$

* Presented to the Society, December 28, 1926.

† Proceedings of the National Academy, vol. 3 (1917), pp. 656–659.

be a second normalized biorthogonal system which is such that (a) *the series*

$$\sum_{n=0}^{\infty}[u_n(x) - \bar{u}_n(x)]v_n(y), \qquad a \leqq x, y \leqq b, \tag{3}$$

converges to a function $H(x, y)$ less than $1/(b-a)$ in absolute value; and

(b) *the convergence is such that the series upon being multiplied by any continuous function may be integrated term by term as to x to yield a uniformly convergent series.*

*Then if the set $\{u_n(x)\}$ involves a k-fold lack of closure on the interval (a, b) the lack of closure of the set $\{\bar{u}_n(x)\}$ is at most k-fold.**

By hypothesis there exist k linearly independent continuous functions $\phi_i(x)$, $i=1, 2, \cdots, k$, which are orthogonal to all functions of the set $\{u_n(x)\}$, and every continuous function $\Phi(x)$, which is orthogonal to all functions $u_n(x)$, is expressible in the form

$$\Phi(x) = \sum_{i=1}^{k} c_i\phi_i(x).$$

It is clear that the entire set

$$\phi_1(x), \phi_2(x), \cdots, \phi_k(x), \ u_0(x), u_1(x), \cdots \tag{4}$$

is closed.

Suppose now that there exist $(k+1)$ continuous functions $f_i(x)$, $i=1, 2, \cdots, k+1$, which are orthogonal to all the functions $\bar{u}_n(x)$. Then set

$$F(x) = \sum_{i=1}^{k+1}\alpha_i f_i(x), \tag{5}$$

the $(k+1)$ constants α_i being chosen not all zero and so as to satisfy the k linear relations

* A slightly different generalization of Birkhoff's theorem to biorthogonal systems in the case $k=0$ (closure) is given by the author, Transactions of this Society, vol. 28, p. 585.

$$\int_a^b \Psi(y)\phi_i(y)dy = 0, \quad (i = 1,2, \cdots, k), \tag{6}$$

where

$$\Psi(y) = F(y) - \int_a^b F(x)H(x,y)dx. \tag{7}$$

Because of hypothesis (b), the function $\Psi(y)$ is continuous. Substituting for $H(x, y)$ the series (3), and observing from (5) that $F(x)$ is orthogonal to all functions $\bar{u}_n(x)$, we obtain from (7) the relation

$$\int_a^b \Psi(y)u_n(y)dy = \int_a^b F(y)u_n(y)dy - \sum_{j=0}^{\infty} \int_a^b F(x)u_j(x)dx \int_a^b v_j(y)u_n(y)dy.$$

Since the system (1) is normalized biorthogonal, however, the right member of this equation vanishes for all n. From this fact and relation (6), it is seen that $\Psi(x)$ is orthogonal to every function of the closed set (4). It follows that $\Psi(y) \equiv 0$, that is,

$$F(y) = \int_a^b F(x)H(x,y)dx.$$

If, now, the maximum numerical value of $F(y)$ on (a, b) is F, and if this value is taken on for $y = y_0$, we have

$$\left| \int_a^b F(x)H(x,y_0)dx \right| = F.$$

But since $|H(x, y)| < 1/(b-a)$, this is impossible unless $F = 0$, and therefore $F(x) \equiv 0$. Hence the $(k+1)$ functions $f_i(x)$ must be linearly dependent and the set $\{\bar{u}_n(x)\}$ involves at most a k-fold lack of closure.

2. *A Theorem in Integral Equations.* The theory of the integral equation with a symmetric kernel contains a theorem to the effect that if the system of characteristic functions

$\{u_n(x)\}$ is not closed then every function which is orthogonal to $u_n(x)$ for all n must be also orthogonal to the kernel.* The following theorem is in a sense analogous for the theory of the integral equation with a general kernel. We consider the integral equation

$$u(x) = \lambda \int_a^b K(x,\xi)u(\xi)d\xi \tag{8}$$

in which the kernel is bounded and has its discontinuities regularly distributed.†

THEOREM II. *If the set of characteristic functions* $\{u_n(x)\}$ *of equation* (8) *is not closed but involves only a finite lack of closure, then there always exists at least one continuous function* $F(x) \not\equiv 0$ *which is orthogonal to the kernel, that is,*

$$\int_a^b K(x,\xi)F(x)dx = 0.$$

By hypothesis there exist k linearly independent continuous functions $f_i(x)$, $i = 1, 2, \cdots, k$, such that

$$\int_a^b f_i(\xi)u_n(\xi)d\xi = 0 \tag{9}$$

for all values of n; and every function orthogonal to all functions $u_n(x)$ is expressible linearly in terms of the functions $f_i(x)$. Consider the function

$$\theta_i(x) = \int_a^b K(\xi,x)f_i(\xi)d\xi. \tag{10}$$

Multiplying it by $u_n(x)$ and integrating, we find because of (8) and (9) that $\theta_i(x)$ is orthogonal to $u_n(x)$ for all values of n. Hence

$$\int_a^b K(\xi,x)f_i(\xi)d\xi = \sum_{j=1}^{k} c_{ij}f_j(x). \tag{11}$$

* T. Lalesco, *Théorie des Équations Intégrales*, Paris, 1912, p. 69.

† The reader will have no difficulty in observing where restrictions of this kind can be lightened if desired.

Let the function $F(x)$ be defined now by the relation

$$F(x) = \sum_{i=1}^{k} \alpha_i f_i(x). \tag{12}$$

Then, from (11), we have

$$\int_a^b K(\xi, x)F(\xi)d\xi = \sum_{j=1}^{k} f_j(x) \sum_{i=1}^{k} c_{ij}\alpha_i. \tag{13}$$

Now the equation

$$\begin{vmatrix} c_{11}-\rho & c_{21} & \cdots & c_{k1} \\ c_{12} & c_{22}-\rho & \cdots & c_{k2} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ c_{1k} & c_{2k} & \cdots & c_{kk}-\rho \end{vmatrix} = 0$$

admits of at least one root $\rho=\rho_1$; hence we may choose the constants α_i in (12) not all zero, and so that

$$\sum_{i=1}^{k} c_{ij}\alpha_i = \rho_1\alpha_j.$$

Substituting this in (13), we find that

$$\int_a^b K(\xi, x)F(\xi)d\xi = \rho_1 F(x). \tag{14}$$

Suppose now that $\rho_1 \neq 0$. Then $F(x)$ is a characteristic function of the associated equation for $\lambda=1/\rho_1$, and it follows that

$$F(x) = \sum \beta_l v_l(x), \tag{15}$$

where the coefficients on the right are constants and the sum includes all the characteristic functions $v_l(x)$ of the associated equation which occur in the biorthogonal normal system $\{u_n(x), v_n(x)\}$, and which correspond to the value $\lambda=1/\rho_1$. From (12), however, $F(x)$ is orthogonal to all

functions of the set $\{u_n(x)\}$, and $F(x) \not\equiv 0$ since the functions $f_i(x)$ are linearly independent. This contradicts (15), since the system $\{u_n(x), v_n(x)\}$ is normalized biorthogonal. It follows that $\rho_1 = 0$, which proves the theorem.

3. *A more General Form of Theorem* II. For simplicity in the formulation of Theorem III we shall call a function $u(x)$ which satisfies the relation

$$\int_a^b K(x,\xi)u(\xi)d\xi = 0$$

a *characteristic function* of equation (8) for the value $\lambda = \infty$. If the equation (8) admits of m linearly independent characteristic functions $u_{-i}(x)$, $i = 1, 2, \cdots, m$, for $\lambda = \infty$, and the associated equation also admits of m characteristic functions $v_{-i}(x)$, $i = 1, 2, \cdots, m$, for $\lambda = \infty$, the system $\{u_{-i}(x), v_{-i}(x)\}$ may be biorthogonalized and normalized provided no linear combination of the functions of either set $u_{-i}(x)$ or $v_{-i}(x)$, is orthogonal to all the functions of the other set*. If the system $\{u_i(x), v_i(x)\}$ is biorthogonal and normal, it follows since

$$\int_a^b u_{-i}(x)v_j(x)dx = \lambda_j \int_a^b v_j(\xi)\int_a^b K(\xi,x)u_{-i}(x)dxd\xi = 0,$$

that the entire system

$$\begin{cases} u_{-m}(x), \cdots, u_{-1}(x), u_0(x), u_1(x), \cdots, \\ v_{-m}(x), \cdots, v_{-1}(x), v_0(x), v_1(x), \cdots, \end{cases}$$

is biorthogonal and normal. We shall call this the extended system $\{u_n(x), v_n(x)\}$, and shall call the entire set of functions $u_n(x)$, $n = -m, (-m+1), \cdots$ the extended set $\{u_n(x)\}$.

THEOREM III. *If the integral equation* (8) *and its associated equation admit as solutions for* $\lambda = \infty$ *the functions of a biorthogonal normal system* $\{u_{-i}(x), v_{-i}(x)\}$, *and if the*

* Goursat, *Cours d'Analyse Mathématique*, 3d ed. (1923), vol. 3, p. 393.

extended set $\{u_n(x)\}$ *is not closed but involves only a finite lack of closure, then there always exists at least one continuous function* $F(x) \not\equiv 0$ *which is linearly independent of the functions* $v_{-i}(x)$, *and which is orthogonal to the kernel, that is,*

$$\int_a^b K(x,\xi)F(x)dx = 0.$$

By hypothesis there exist k functions $f_i(x)$, $i=1, 2, \cdots, k$, which satisfy (9), where n runs over all values in the extended set $\{u_n(x)\}$. Precisely as in the preceding proof it follows that $\theta_i(x)$ in (10) is orthogonal to the functions $u_n(x)$ for $n=0, 1, 2, \cdots$. But further

$$\int_a^b \theta_i(x)u_{-j}(x)dx = \int_a^b f_i(\xi)\int_a^b K(\xi,x)u_{-j}(x)dxd\xi = 0,$$

since $u_{-j}(x)$ is a characteristic function for $\lambda=\infty$. Hence again $\theta_i(x)$ is orthogonal to $u_n(x)$ for all n. As in the preceding proof we may construct now a function $F(x)$ by formula (12) which satisfies equation (14). Moreover the previous reasoning shows that $\rho_1=0$, and hence that $F(x)$ is orthogonal to $K(x, \xi)$. To show that $F(x)$ is linearly independent of the functions $v_{-i}(x)$, we assume the contrary, namely that

$$\gamma_0 F(x) + \sum_{i=1}^{m}\gamma_i v_{-1}(x) \equiv 0.$$

Since the extended set $\{u_n(x), v_n(x)\}$ is biorthogonal and normal, while $F(x)$ is orthogonal to every function $u_n(x)$, it follows, upon multiplying the relation by $u_{-j}(x)$ and integrating, that $\gamma_j=0$, $j=1, 2, \cdots, m$. This implies that $F(x)\equiv 0$ which involves a contradiction.

COROLLARY. *If the system of all linearly independent solutions of equation* (8) *and its associated equation for* $\lambda=\infty$ *may be made biorthogonal and normal, and if any subset of the extended set* $\{u_n(x)\}$ *involves only a finite lack of closure, then the extended set* $\{u_n(x)\}$ *is closed.*

If a subset involves only a finite lack of closure, the same must be true of the entire set.

4. *Examples.* The following examples are given to illustrate the necessity of certain parts of the hypotheses which have been made. Thus in Theorem II the case of an infinite lack of closure is excluded. Such a case is represented by the Volterra equation in which $a=0$, $b=1$, and

$$K(x,\xi) = \begin{cases} 0 \text{ for } \xi > x, \\ 1 \text{ for } \xi \leqq x. \end{cases}$$

The equation admits of no characteristic values other than 0, so the lack of closure is infinite. Since

$$\int_0^1 K(x,\xi)f(x)dx = \int_0^x f(x)dx = 0$$

implies $f(x) \equiv 0$, the assertion of the theorem clearly does not hold.

In the Corollary to Theorem III, the case in which the totality of solutions for $\lambda = \infty$ cannot be made biorthogonal is excluded. This case is illustrated by the example*

$$K(x,\xi) = K_1(x,\xi) + K_2(x,\xi),$$

where

$$K_1(x,\xi) = \phi_1(x)\phi_2(\xi),$$

$$K_2(x,\xi) = \sum_{n=3}^{\infty} \frac{\phi_n(x)\phi_n(\xi)}{\lambda_n},$$

the set $\{\phi_n(x)\}$ being any closed normalized orthogonal set and the series for $K_2(x, \xi)$ converging uniformly. The functions $K_1(x, \xi)$ and $K_2(x, \xi)$ are orthogonal, and hence the set of characteristic functions for $K(x, \xi)$ is composed of those for $K_1(x, \xi)$ and $K_2(x, \xi)$.† The characteristic functions for $K_2(x, \xi)$ are obviously $\phi_n(x)$, $n=3, 4, \cdots$. On the other hand equation

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* This example was given to me by Professor J. Tamarkin of Dartmouth College.

† Goursat, loc. cit., p. 402.

$$u(x) = \lambda \int_a^b K_1(x,\xi)u(\xi)d\xi$$

is of the form

$$u(x) = \lambda \int_a^b \phi_2(\xi)u(\xi)d\xi\phi_1(x) = c\phi_1(x),$$

whereas the substitution of this value in the equation yields

$$c\phi_1(x) = \lambda c\phi_1(x) \int_a^b \phi_2(\xi)\phi_1(\xi)d\xi = 0.$$

Hence $k_1(x, \xi)$ admits of no characteristic functions. Further if $f(x)$ is a characteristic function for $\lambda = \infty$ we have upon setting

$$\int_a^b f(\xi)\phi_1(\xi)d\xi = f_i$$

the relation

$$\int_a^b K(x,\xi)f(\xi)d\xi = f_2\phi_1(x) + \sum_{n=3}^{\infty} \frac{f_n\phi_n(x)}{\lambda_n} = 0.$$

Multiplying this by $\phi_j(x)$ and integrating we find that $f_j = 0$ for $j = 2, 3, \cdots$, whence $f(x)$ must be of the form $f(x) = c\phi_1(x)$ and $\phi_1(x)$ is the only characteristic function for $\lambda = \infty$. In similar fashion it is seen that $\phi_2(x)$ is the only characteristic function of the associated equation for $\lambda = \infty$. Since on the one hand $\phi_1(x)$ and $\phi_2(x)$ are orthogonal the hypothesis of the corollary cannot be satisfied. On the other hand since the entire set of characteristic functions

$$\phi_1(x), \ \phi_3(x), \ \phi_4(x), \cdots$$

is not closed while the subset $\phi_i(x)$, $i = 3, 4, \cdots$, involves only a two-fold lack of closure the assertion of the theorem clearly fails.

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A CONNECTED AND CONNECTED IM KLEINEN POINT SET WHICH CONTAINS NO PERFECT SUBSET

BY B. KNASTER AND C. KURATOWSKI

1. *Introduction.* Professor R. L. Moore has shown in this Bulletin (vol. 32, p. 331) that there exist point sets connected and connected im kleinen* which contain no arc. We shall prove in this paper the existence of such a set containing no *perfect* subset. The set has the additional property that it is contained in a *regular curve* (in the sense of K. Menger)†.

2. *The Sierpinski Regular Curve.* Let R be the Sierpinski regular curve,‡ defined as follows. Let T be an equilateral triangle. Divide T in 4 equal triangles. Let T_0, T_1, T_2 denote those three triangles which have a common vertex with T. Similarly divide each of the triangles T_0, T_1, T_2 in 4 equal triangles and let T_{00}, T_{01}, T_{02}, T_{10}, $\cdots$, T_{22} denote those having a common vertex with T_0 or T_1 or T_2; and so on *ad inf.*

The point set formed by the boundaries of all the triangles $T_{\alpha_1\alpha_2\cdots\alpha_i}(\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_i = 0, 1, \text{or } 2)$ and all their limit points is the regular curve R.

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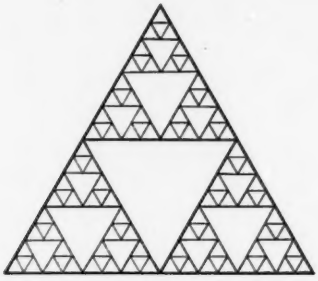
* A point set M is said to be *connected im kleinen* (or to be regular) if, for every point p and every positive number e, there exists a positive number d such that if x is any point of M at a distance from p less than d then x and p both lie in some connected subset of M of diameter less than e.

† A continuum C is called a regular curve if, for every positive number e, C can be expressed as the sum of a finite number of continua each of diameter less than e, each pair of continua having at most a finite number of points in common (see K. Menger, Mathematische Annalen, vol. 95 (1925), p. 300). Every regular curve is a continuous curve whose every subcontinuum is a continuous curve (see H. M. Gehman, Annals of Mathematics, vol. 27 (1925), p. 42).

‡ Prace Matematyczno-Fizyczne, vol. 27 (1915). The Sierpinski curve R contains three points of degree 2, a countable set of points of degree 4, the remainder being composed of points of degree 3. See also Comptes Rendus, vol. 160 (1915), p. 302.

3. *Properties of the Sierpinski Curve.* The complementary set of R is composed of a countable set of regions. Let I_0 denote the unbounded region (exterior to T) and $I_1, I_2, \cdots, I_n, \cdots$ the bounded regions (for $n>0$, I_n forms the interior of a triangle). Let B_n denote the boundary of I_n and V the countable point set of all the vertices of the triangles $T_{\alpha_1\alpha_2\cdots\alpha_k}$. The following properties of the curve R are to be noticed:



PROPERTY 1. *Given a positive integer k, no point of R belongs to more than two of the triangles $T_{\alpha_1\alpha_2\cdots\alpha_k}$.*

PROPERTY 2. *If $m\neq n$, then $\bar{I}_m \cdot I_n \subset V$*.*

PROPERTY 3. *Given two points a and b of R and an index n, there exists a subcontinuum C of R containing a and b and such that $C \cdot B_n \subset V+a+b$.*

In a similar manner we have

PROPERTY 4. *Given a triangle $T_{\alpha_1\alpha_2\cdots\alpha_k}$ and an index n, any two points a and b of $R \cdot T_{\alpha_1\alpha_2\cdots\alpha_k}$ may be joined by a subcontinuum C of $R \cdot T_{\alpha_1\alpha_2\cdots\alpha_k}$ such that $C \cdot B_n \subset V+a+b$.*

4. *Two Lemmas.* We shall now prove two lemmas.

LEMMA I. *If K is a bounded continuum such that $K \cdot V=0$ and there exists no index n such that $K \subset \bar{I}_n$, then the set $K \cdot R$ contains a perfect subset.*

* The symbol $\bar{X}$ denotes the set $X+$ all its limit points. The symbol $X \subset Y$ means that X is contained in Y.

PROOF. It is evident that

$$K = K \cdot R + K \cdot I_0 + K \cdot I_1 + \cdots + K \cdot I_n + \cdots ;$$

hence

$$(1) \quad K = K \cdot R + K \cdot I_0 + K \cdot I_1 + \cdots + K \cdot I_n + \cdots .$$

Suppose $K \cdot R$ does not contain any perfect subset. As $K \cdot R$ is closed, it follows that $K \cdot R$ is a finite or countable point set.

Let S_n denote the set $K \cdot I_n$ and let Q denote the set $K \cdot R - (S_0 + S_1 + \cdots + S_n + \cdots)$. Hence Q is a finite or countable set of points: $p_1, p_2, \cdots$. It follows by (1) that

$$(2) \quad K = p_1 + p_2 + \cdots + S_0 + S_1 + \cdots + S_n + \cdots .$$

The right-hand side of the identity (2) is composed of mutually exclusive sets, since by Property 2, if $m \neq n$, then $K \cdot I_m \cdot K \cdot I_n \subset K \cdot V = 0$. But this contradicts a theorem of Sierpinski's* to the effect that if K is a sum of a finite ($\geqq 2$) or countable number of mutually exclusive closed point sets, then K is not a bounded continuum. Thus the existence of a perfect subset of $K \cdot R$ is established.

LEMMA II. *If Z is a subset of R such that each perfect subset of R contains a point belonging to Z, then $Z+V$ is connected and connected im kleinen.*

PROOF. Suppose that the set $Z+V$ is not connected. Then† there exist two points a and b of $Z+V$ and a bounded continuum K which separates a from b and is such that $K \cdot (Z+V) = 0$. Therefore

$$(3) \quad K \cdot (V + a + b) = 0.$$

Since $K \cdot Z = 0$, it follows, by hypothesis, that the set $K \cdot R$ does not contain any perfect subset. By Lemma I, there exists an index n such that $K \subset I_n$. As $I_n = I_n + B_n$ and $R \cdot I_n = 0$, it follows that

* Tôhoku Mathematical Journal, vol. 13 (1918), p. 300.

† See our paper *Sur les ensembles connexes*, Fundamenta Mathematicae, vol. 2 (1921), p. 233, Theorem 37.

$$K \cdot R \subset B_n. \tag{4}$$

By Property 3, the points a and b may be joined by a subcontinuum C of R such that

$$C \cdot B_n \subset V + a + b. \tag{5}$$

Since $K \cdot C = K \cdot R \cdot C$, it follows from (4) and (5) that

$$K \cdot C \subset B_n \cdot C \subset V + a + b.$$

Hence by (3) $K \cdot C = 0$, contrary to the assumption that K separates a from b. Thus the supposition that $Z+V$ is not a connected point set leads to a contradiction.

Now let H denote any one of the triangles $T_{a_1 a_2 \cdots a_k}$. Each perfect subset of $R \cdot H$ has a common point with $Z \cdot H$. By an argument similar to that used above it may be proved (with the help of Property 4 instead of Property 3) that the set $(Z+V) \cdot H$ is connected. It follows by Property 1 that $Z+V$ is connected im kleinen.

5. *Conclusion.* We may now state the following theorem.

THEOREM. *There exists in the regular curve R of Sierpinski's a connected and connected im kleinen point set which contains no perfect subset.*

PROOF. By a theorem due to F. Bernstein* the plane may be decomposed into two mutually exclusive subsets E and F such that each perfect set contains points of both of them. It follows that each perfect subset of R has a common point with $E \cdot R$. By Lemma II the set $M = E \cdot R + V$ is connected and connected im kleinen.

The set M contains no perfect subset. For suppose P is a perfect subset of M. As the set V is countable, $P - V$ contains a perfect subset P_1. Hence $P_1 \subset E$, contrary to the assumption that P_1 has a common point with F. Thus M is a connected and connected im kleinen point set which contains no perfect subset.

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* Leipziger Berichte, vol. 60 (1908).

SCHLESINGER ON LEBESGUE INTEGRALS

Lebesguesche Integrale und Fouriersche Reihen. By L. Schlesinger and A. Plessner. Berlin and Leipzig, Walter de Gruyter Company, 1926. viii+229 pp.

This book is an outgrowth of a course of lectures by Professor L. Schlesinger at the University of Giessen in the winter semester of 1921–22 and was written with the cooperation of Dr. A. Plessner, who was at that time a student in the course, and later Professor Schlesinger's assistant. As the title indicates the subject is Lebesgue integrals and Fourier expansions from the point of view of Lebesgue's definition of integration.

Necessarily the first part is a study of the properties of real functions defined on point sets in a space of m dimensions. The first five chapters or 180 pages form a masterly exposition of the essential facts of real variable theory, particularly integration, and the sixth chapter of forty-two pages deals with Fourier series.

The headings give a sufficiently definite idea of the contents of each chapter. I. Fundamental concepts of points sets; II. Measure of point sets; III. Concerning functions of real variables; IV. The Lebesgue integral; V. Functions of one and two variables; VI. Fourier series.

Particular attention is directed to Chapter II in which the Borel-Lebesgue definition of measure is developed and compared with the older Peano-Jordan definition of content, and to Chapter IV where the Lebesgue integral is defined following Lebesgue's original geometric point of view. In the last section of this chapter (p. 129 ff.) attention is called to other useful definitions of the L-integral, due to Lebesgue himself, W. H. Young, J. Pierpont, and F. Riesz. This last definition will be discussed by the reviewer a little further on in greater detail, but at this point, it is well to note that mention should have been made along side of these of a two-way infinite series definition of a Lebesgue integral due to M. B. Porter.* By means of this definition the properties of the L-integral and the theorems of Lusin, Egoroff, and other important results are almost immediate.

Chapter VI contains an elegant and concise treatment of the most important newer results for Fourier series, notably those beginning with Lebesgue's *Leçons sur les Séries Trigonométriques*, 1906, which include the classic results of Riemann, Dini, Lipschitz, Dirichlet, Jordan, and others. The chapter closes with theorems on summability and convergence in the mean of Fejér, Cesàro, E. Fischer, and F. Riesz.

At the close of the preface is found a list of references to monographs, treatises, and encyclopedia articles whose content is closely related to the present text. Scattered throughout the book are to be found in footnotes, full references to the original memoirs bearing on all phases of the field.

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* This Bulletin, vol. 28 (1922), pp. 105–8.

The feature which, aside from the excellent form of the entire book, seems of foremost importance to the writer, receives no notice whatever in the table of contents and but one citation in the index. The number of references, however, is, in reality, six. These occur on pages 114, 123, 131, 194, 215, 221. This feature is the use of a sequence of step-functions (*Treppenfunktionen*) or *horizontal functions* in certain questions of convergence. The reviewer* has given the following definition:

A *horizontal function*, $h_n(x)$, *of index* n on an interval $I \equiv a \leqq x \leqq b$, associated with a subdivision of I into n subintervals, is one such that in the interior of the ith subinterval $h_n(x)$ is constant, i.e., $h_n(x) \equiv h_{in}$. The function need not be defined at the end points of the subintervals. Clearly $h_n(x)$ is represented for each value of n by n horizontal segments. Such functions are implicit in Riemann's and Lebesgue's (analytic) definition of an integral. In their explicit use, F. Riesz who called them *"fonctions simples"* seems to have been the first (references in the footnote, p. 131, in the present book under review). His use of them to give an alternate definition of an L-integral will be stated.

Concerning these horizontal functions, the following theorems are immediate. They will be stated without proof.

THEOREM I. *For every continuous function $f(x)$ on an interval I, there exists a sequence of horizontal functions $f_n(x)$ which has $f(x)$ for its limit function for every x in I. Furthermore, the sequence can be chosen so that the approach to the limit function will be monotone and uniform.*

THEOREM II. *For every bounded Riemann integrable function $f(x)$ on I there exists a sequence of horizontal functions $f_n(x)$ which has $f(x)$ for its limit function for every x in I save for a null set. Furthermore, the sequence can be chosen so that the approach to the limit function will be monotone.*

In the above two theorems, the set of horizontal functions may be built up in terms of a value of $f(x)$ in each subinterval. In particular, the maximum value (upper limit) or the minimum value (lower limit) may be used in each subinterval. We shall prove the following theorem used by Schlesinger for the simplest case.

THEOREM III. *For every bounded measurable function $f(x)$ on I, there exists a sequence of uniformly bounded horizontal functions $f_n(x)$ which has $f(x)$ for its limit function for every value of x on I save a null set.*

Proof. By hypothesis $f(x)$ has an indefinite L-integral

$$F(x) = \int_a^x f(t)\, dt,$$

where $F(x)$ has a bounded incremental ratio on I and $F'(x) = f(x)$ almost everywhere on I. Let $x_{in} = i(b-a)/n$ and let N be the number of sub-

* H. J. Ettlinger, *Note on a fundamental lemma concerning the limit of a sum*, this Bulletin, vol. 32 (1926), p. 69. See also H. J. Ettlinger, *On multiple iterated integrals*, American Journal, vol. 48 (1926), pp. 215-222.

intervals defined for every value of n by each pair of consecutive points $x_{ij}, i=1, 2, \cdots, j; j=1, 2, \cdots, n$. Let

$h_N(x)=[F(x_{i+1,N})-F(x_{iN})]/[x_{i+1,N}-x_{iN}]$. Then $\lim_{n\to\infty} h_N(x)=f(x)$

for every x on I save a null set. Schlesinger proves this theorem for a bounded measurable function of m variables defined on a measurable point set.

The following theorem is in effect the equivalent of Riesz's definition of an L-integral. It may be regarded as a converse of Theorem III.

THEOREM IV. *If a uniformly bounded sequence of horizontal functions $f_n(x)$ on I has a limit function $f(x)$ almost everywhere on I, then $f(x)$ is bounded and measurable on I, i.e., is L-integrable on I.*

In terms of horizontal functions it is easy to establish the substantial equivalence of a *principle* used by the reviewer* on various occasions and named the Duhamel-Moore theorem,† and a theorem due to Lebesgue‡ on the integral of the limit of a sequence of functions. The Duhamel-Moore theorem (DM) may be stated for an interval I as follows:

HYPOTHESIS. 1. *The set of n non-overlapping subintervals I_{in}, $i=1, 2, \cdots, n$, of length l_{in} are obtained by subdividing the interval I. The sets of numbers r_{in}, r'_{in} are such that $|r_{in}-r'_{in}|$ is uniformly bounded.*

2. *If P is a point of I then $\lim_{n\to\infty}(r_{iP_n n}-r'_{iP_n n})=0$ almost everywhere on I.*

3. $\operatorname{Lim}_{n\to\infty}\sum_1^n r_{in}l_{in}$ *exists.*

CONCLUSION

1. $\operatorname{Lim}_{n\to\infty}\sum_1^n r'_{in}l_{in}$ *exists.*

2. $\operatorname{Lim}_{n\to\infty}\sum_1^n r_{in}l_{in}=\operatorname{Lim}_{n\to\infty}\sum_1^n r'_{in}l_{in}$.

Lebesgue's theorem (L) may be stated as follows:

HYPOTHESIS. 1. *$f_n(x)$ is a uniformly bounded sequence of integrable functions on I.*

2. $\operatorname{Lim}_{n\to\infty} f_n(x)=f(x)$ *almost everywhere on I.*

CONCLUSION

1. *$f(x)$ is bounded and integrable on I.*

2. $\operatorname{Lim}_{n\to\infty}\int_a^b f_n(t)dt=\int_a^b f(t)dt$.

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* See my paper entitled *On multiple iterated integrals*, loc. cit., for complete references.

† R. L. Moore, *On Duhamel's theorem*, Annals of Mathematics, (2), vol. 13 (1912), pp. 162–3.

‡ *Leçons sur l'Intégration*, etc. (Borel monograph.) Paris, Gauthier-Villars, 1904, p. 114.

By the use of Theorems III and IV, we derive (DM) from (L). Let $r_{in}-r'_{in}$ define a sequence of horizontal functions on I. The limit function is obviously a null function whose integral is zero. Conclusions 1 and 2 of (DM) follow immediately from (L).

Conversely we derive (L) from (DM). By Theorem IV, $f(x)$ is bounded and integrable in I. Let $h_n(x)$ be a set of horizontal functions on I with respect to I_{in} approaching $f(x)$ as a limit function almost everywhere on I. Identify $r_{in}=h_{in}$. Also let $H_n^{(k)}(x)$ be a horizontal function of index n on I associated with I_{in} and having $f_k(x)$ as a limit function almost everywhere on I. Identify $r'_{in}=H_{in}^{(n,k)}$ and Conclusion 2 of (L) follows from (DM).

It is worthwhile emphasizing the role which horizontal functions, together with the principle isolated by R. L. Moore, are destined to play in the theory of functions of a real variable. They may be made the basis for a concise and elegant treatment of the greater part of the theory. The bulk of a book like Hobson's *Theory of Functions of a Real Variable*, volume I, may be reduced to one-third or less by their use. Schlesinger in the book under review has made a beginning in this direction.

H. J. ETTLINGER

FUBINI AND ČECH, PROJECTIVE DIFFERENTIAL GEOMETRY

Geometria Proiettiva Differenziale, Vol. I. By G. Fubini and E. Čech. Bologna, N. Zanichelli, 1926. 388 pp.

"Un nuovo indizio dei sentimenti fraterni che vanno sempre più legando fra loro i vari rami della matematica!"

These words of Segre were chosen by Wilczynski to be inscribed on the title page of his prize memoir *Sur la théorie générale des congruences*. Thus the founder of the American school of projective differential geometry indicated that he was studying the projective differential properties of a geometric configuration by the means of the invariants and covariants of a completely integrable system of linear homogeneous partial differential equations under a certain continuous group of transformations, in the sense of Lie. The same sentiment would be no less appropriate as a motto for the new book by Fubini and Čech, since these distinguished protagonists of the Italian school of projective differential geometry define a configuration by means of differential forms, after the manner of Gauss, and employ the absolute calculus of Ricci.

Those who know the absolute calculus only as it is used in the theory of relativity will be interested to see this geometric application of it. And those who know only Wilczynski's method of attacking a problem in projective differential geometry will be eager to learn this new theory. Wilczynski's method is particularly adapted to certain types of problems and has a power and elegance of its own. But it has some inconveniences. For instance, certain calculations become quite laborious, which are accomplished more easily and efficiently by the tensor analysis.

The first volume of the treatise before us is dedicated to the dean of differential geometers, Luigi Bianchi, and is devoted to the geometry of

curves and surfaces in space of three dimensions. The projective definition of a surface by means of three differential forms of the first order, two quadratic and one cubic, is due to Fubini. Of these, one, F_3, vanishes for the asymptotics, and another, F_3, which is apolar to F_2, vanishes for the curves of Darboux. The ratio $F_3 : F_2$ is called the *linear projective element.* For this Bompiani has furnished a geometric interpretation; Čech has given an algebraic theory of the invariants thereof. The extremals of the integral $\int F_3 : F_2$ are called *pangeodesics.*

Fubini has defined a *projective applicability,* two surfaces being projectively applicable in case they have the same linear projective element. Moreover, for a non-ruled surface, Fubini has obtained a normalization of the proportionality factor of the homogeneous coordinate system such that his *normal coordinates,* if the asymptotic net is parametric, satisfy a certain canonical system of differential equations of the same form in covariant derivatives as Wilczynski's much used canonical equations in ordinary derivatives. Introducing the normalized form of F_2 as a projective substitute for the squared arc element, or first form of Gauss, Fubini defines a *projective normal,* which G. M. Green also discovered independently, projective geodesics, projective lines of curvature, etc. For this projective metric of Fubini, a geometric interpretation was furnished by Wilczynski.

The principle of duality plays its proper role, particularly in the formulation of the differential equations for the determination of a surface when the three forms that define it are given. The authors prefer, when using a local coordinate system at a point of a surface, to define the system so that a point and a plane whose corresponding local coordinates are equal are respectively pole and polar with respect to the quadric of Lie, called by Wilczynski the osculating quadric. With this convention the condition of united position of point and plane does not have its usual simple form.

An entire chapter is devoted to the theory of space curves, another to the theory of ruled surfaces, and still another to the transformation of surfaces by means of Weingarten congruences. The calculation of the integrability conditions for a curved surface in arbitrary parameters, by means of covariant derivatives, deserves special mention, as does also the treatment of various interesting classes of surfaces.

The book contains surprisingly few typographical errors, and these are for the most part easily corrected. One might wish that the type setter had been consistent in his use of italics for mathematical symbols. Occasionally the differentials du and dv and some other symbols are not in italics. The notation is well chosen, although one of my students expressed the wish that the authors had used c_{rst} instead of a_{rst} for the coefficients of F_3, since they had already used a_{rs} for the coefficients of F_2. The style of exposition is clear, concise, and direct.

This treatise fills a long felt want. There is no other of its kind. Hereafter the projective differential geometer must know his Fubini and Čech, and every student of geometry has now a new reason for acquiring at least a reading knowledge of *la lingua italiana.*

E. P. LANE

HOBSON'S SECOND VOLUME

The Theory of Functions of a Real Variable and the Theory of Fourier's Series. By E. W. Hobson. Second Edition, vol. II. Cambridge University Press, 1926. x+780 pp.

It is a long time since I previously had so much delight in examining a book as I have recently experienced in the rapid reading of the second volume of Hobson's *Functions of a Real Variable*. It is characterized by an extraordinary richness of content and by the remarkable indications which it affords of the marvelous vitality and fecundity of the current investigations in the general theory of functions of a real variable and in the theory of Fourier's series in particular. This second volume completes the author's extension and revision of the one-volume work which first appeared in 1907, the earlier part of his revision having been published in 1921 as volume I of the second edition. Almost the whole of the matter for the new volume II has been re-written and much new matter has been added so that the extent of the whole work is now twice as great as that of the first edition. The new matter is largely the fruit of investigations carried out in the last twenty years. The completed work now affords a most effective testimonial to current progress in one division of mathematics as well as the most important exposition yet given of knowledge in that domain.

The volume is beautifully printed in the style for which the Cambridge University Press has justly become famous among mathematicians. The volume is bound so as to be pleasing both to the eye and to the hand when new; but, as experience with earlier books so bound has shown, it has not the durability which one has a right to expect in the case of a volume of such permanent value and one likely to be so frequently used. In some places the author might have improved the appearance of the page by a happier choice of notation or a more suitable arrangement of formulas. A considerable number of minor errors escaped detection in the final proof-reading. But where there is so much excellence in regard to the matters of major importance, it is perhaps ungracious to do so much as to call attention to these minor blemishes.

There can be no doubt that the two volumes of the second edition of this work will be for a long time of constant use to every one who has to do with investigations touching the theory of functions of a real variable and the theory of Fourier series.

The first of the ten chapters of volume II deals with sequences and series of numbers. Into the 98 pages of this chapter the author compresses an excellent exposition of the theory of series with real constant terms, the earlier part of it being abbreviated by the use of some theorems from volume I. The reader will be interested in the compactness of the exposition and the comprehensiveness of the theory so far as the problem of convergence is concerned. It is in no sense a complete summary of what is known about series with constant real terms but it is an account which sets all the main facts in their proper light. Nearly forty pages of the chapter are given to the problem of the summability of series of constant terms.

In this part no effort is made to reach a comprehensive exposition. In fact attention is directed chiefly to an account of some of the main features of summability by the methods of Cesàro, Hölder, and Riesz; and the more general problems connected with the summability of series find here no systematic treatment.

An exhaustive exposition of the general theory of the summability of series has not yet appeared. The theory has now probably attained such a state of development as would render feasible such an exposition. It is one of the important desiderata which must be felt by every one who has to deal with the more delicate questions concerning the convergence of infinite processes. Such a work, if properly prepared, would be of constant use to a large number of investigators in many fields of analysis.

In Chapter II (pp. 99–171) is given a systematic account of the theories of convergence and oscillation of sequences and series of functions of one or more variables. This is done in an elegant and pleasing way and with such a comprehensiveness as one would expect in a general treatise. Special attention is given to the problems associated with uniform convergence and with the various modifications of the notion of uniformity of convergence. The question of the continuity of the sum-function of a series is treated in detail.

An application of the general results of Chapter II is made in Chapter III (pp. 172–227) to the special but important case of power series. Problems relating to interior points of the interval of convergence are disposed of briefly. Much attention is devoted to the behavior of the sum-function near the ends of the interval of convergence and to its relation to the series obtained by giving to the variable its value at an end-point of this interval. In this connection one finds an interesting and illuminating account of the Tauberian theorems which have been the subject of a considerable amount of recent investigation.

Chapter IV (pp. 228–288) contains an account of the theorem of Weierstrass concerning the representation of a continuous function (of one or more variables) by means of sequences of polynomials; of the theory of convergence of sequences on the average; of F. Riesz' classification of summable functions; and of Baire's classification of functions. The fundamental result of Baire, relating to the representation of a function as the limit of a sequence of continuous functions, is proved by a method due to de la Vallée Poussin, but with some modification and extension.

Chapter V (pp. 289–388) is on sequences of integrals. It contains an exposition of those parts of the theory of integration which were not already treated in volume I. In particular, considerable space is given to the theories of integration due to W. H. Young, to Tonelli, and to Perron. The chapter concludes with a very brief account of the summability of integrals, especially by the methods analogous to Cesàro and Hölder summability of series. The author expresses (p. 384) the opinion that Perron's recent definition of an integral may prove to be of great importance in the future developments of the concept of integration.

In Chapter VI (pp. 389–421) is to be found an account of the methods of construction of functions exhibiting assigned peculiarities of behavior

and in particular of functions which are continuous but non-differentiable. An exposition is given of various methods of condensation of singularities.

Chapter VII (pp. 422–475) is devoted to the problem of the representation of functions as limits of integrals. A special feature of this chapter, and indeed of the whole volume, is the prominence which is here given to what the author calls the General Convergence Theorem and to its developments and consequences. This general theorem is treated fully, and with considerable novelty of exposition, in Chapter VII, and various applications of it to Fourier's series and integrals are given in the following chapters.

The remaining three chapters of this volume correspond to the final chapter of the first edition.

Chapter VIII (pp. 476–719) deals with trigonometric series. It is by far the longest chapter in the volume. It is here that we find the amplest evidence of the recent activity of investigators to be found anywhere in the volume,—a volume which is marked throughout with much evidence of rapid recent progress. Most of the recent progress in the development of the theory of trigonometric series has been due to the use of two recently constructed tools—that afforded by the theory of Lebesgue integration and various other more recent definitions of integration and that due to various definitions of the summability of series recently exploited with so much vigor. Though trigonometric series have had a remarkable history covering more than a century and a half, the extraordinary recent progress in our knowledge of them and the questions that still remain unanswered show unmistakably that the subject is capable of much further development.

The theory of trigonometric series has had a nearly continuous and remarkably intimate connection with the development of much of the general theory of functions of a real variable. This alone is sufficient to justify the author in analyzing it with some fulness in such a general treatise as that under review. The connection began early and its intimacy is reflected in much of the most recent development. Concerning the connection the author says (p. 476): "Historically, the questions which have arisen in connection with this theory have influenced the development of the theory of functions of a real variable to an extent which is comparable with the degree in which the theory of functions in general has been affected by the theory of power series. The theory of sets of points, which led later to the abstract theory of aggregates, arose directly from questions connected with trigonometric series. The precise formulation by Riemann of the conception of the definite integral, and the gradual development of the modern notion of a function as existent independently of any special mode of representation by an analytical expression, are further examples of the results of the study of the properties of these series upon Mathematical Analysis."

The chapter contains an adequate treatment both of the earlier and of the later theory of trigonometric series and especially of Fourier series, limitation being made to the case in which the arguments involved in the trigonometric functions are integral multiples of the independent variable.

A reference is made to the fact that there are important trigonometric series in which the arguments are differently determined, but these additional types are not treated. Notwithstanding the great richness of detail in the theory as here expounded, one comes from a reading of the chapter with the feeling that the whole theory is yet in a state of development which is far from complete. It can be predicted that much further development of the theory will take place in the near future, both in the way of deeper penetration into details and in the analysis of general methods of treatment and especially of more general types of summability of the series. The theory of summability is capable of much further development by aid of the theory of functions of a complex variable (and especially the theory of residues), a tool which is nowhere employed in the exposition as given by Hobson.

Chapter IX (pp. 720–752) is devoted to the representation of functions by Fourier integrals, together with the modern theory of Fourier transforms. A significant part of the chapter is given to the summability of Fourier integrals, a subject which appears to be capable of much further development.

The final Chapter X (pp. 753–772) contains a brief treatment of series of normal orthogonal functions, a subject which is included "not only on account of the intrinsic importance of the subject, but also because the processes which have been employed in various recent investigations in this domain afford excellent illustrations of ideas and methods which have been developed earlier in this work." In this chapter no attempt at a comprehensive treatment has been made. In fact the exposition scarcely contains so much as an introduction to this extended subject. Many of the properties of Fourier series can be carried over to certain classes of expansions in orthogonal functions—a fact which is but little more than indicated in the brief account to be found here. Just as the first chapter leaves one with the desire for a full exposition of a subject but partially treated so the final chapter leaves one wishing for an adequate and exhaustive treatise on the theory of expansions in orthogonal and biorthogonal functions. What the author gives is scarcely enough to suggest the vast richness of this subject. In this respect the chapter is different from most of those contained in the volume,—the exposition being in most cases adequate and satisfying.

R. D. CARMICHAEL

SHORTER NOTICES

Versicherungsmathematik. I: *Elemente der Versicherungsrechnung.* By Friederich Boehm. (Sammlung Göschen.) Berlin, W. de Gruyter, 1925. 144 pp.

This is an elementary text on the mathematics of life insurance, the subject being treated particularly from the point of view of German and continental practice. It is divided into four chapters. The first deals with the elements of the theory of compound interest—just enough to carry through the accumulation and discount applications necessary in the theory of life insurance.

The second chapter starts in with the life table and forty pages are devoted to questions concerning expectation of life, life annuities of various kinds and single and annual premiums for ordinary and limited payment life, temporary, and endowment insurance.

The third chapter develops the theory of premium reserves including a discussion of net level premium reserves and Zillmer's formula for reserves; in particular, the full preliminary term method of Zillmer is explained.

Chapter IV deals with annuities and insurance on joint lives, including the calculation of single and annual premiums and reserves. Some little space is given to mixed forms of insurance, widows' pensions prior to remarriage and annuities to orphans.

This little book is evidently intended as an introduction to the theory of life insurance, the formulas being developed in considerable detail and practically every formula immediately illustrated by a numerical example. The author is not familiar with the international code of symbols adopted and carefully adhered to by most actuaries, or else he has intentionally modified these symbols. This is particularly noticeable in the use of the summation sign for the N_x and M_x, the use of the letter R instead of V to designate reserves, and the use of the symbol for endowment insurance for temporary insurance. Similar modifications have been made for the present value and accumulated value compound interest functions. These changes are unfortunate and quite uncalled for and are bound to cause confusion to all students who get their introduction to actuarial theory through this book when they come to read other text books on this subject. The use of the summation sign for the N and M commutation symbols makes the formulas quite cumbersome in appearance. It is to be hoped that if a new edition of this book should appear the author will adopt the symbols of the international code of actuaries. Apart from these criticisms the book is well written and may be regarded as a good introduction to actuarial theory.

J. W. GLOVER

Traité de Balistique Extérieure. By P. Charbonnier. Vol. I. Paris, Gauthier-Villars, 1921. ix+637 pp.

For a number of years, General Charbonnier has been preparing an encyclopedic treatise on exterior ballistics in six volumes, of which this is the first.

The first part of this volume contains the most exhaustive treatment of the trajectory in vacuum which has yet been written.

The second part deals with the rectilinear motion of a particle in a medium where the resistance depends on the velocity alone; the Siacci ballistic functions are introduced and their simplest properties derived.

The third part is concerned with the general case of the trajectory of a heavy particle, the resistance being a function of the velocity. Here, as in the second part, the change in resistance due to the altitude variation in the density of the atmosphere, is neglected. This part begins with a very detailed study of the qualitative properties of the trajectory profusely illustrated by well-drawn diagrams. This investigation is of particular interest to the mathematician, and might be used as a source of examples of the behavior of the integral curves in an advanced course in differential equations. As far as the reviewer has examined the theorems in this section, they are correct, but the proofs are frequently lacking in rigor and occasionally contain actual errors, which are, however, not very difficult to detect and remedy.

This section also contains a brief and very readable exposition by A. Denjoy of Drach's recent determination of all the forms of the resistance function for which the equations of the trajectory are integrable by quadratures.

The further treatment of the trajectory follows the customary lines of the Siacci and similar theories, and the volume ends with a chapter on the use of power series expansions for computing a small arc of the trajectory.

The modern methods in ballistics are not dealt with here, but are reserved for the later volumes, none of which has yet appeared.

T. H. GRONWALL

Séries Analytiques. Sommabilité. (Mémorial des Sciences Mathématiques, No. 7.) By A. Buhl. Paris, Gauthier-Villars, 1925. 55 pp.

This monograph deals with analytic extension by means of various methods of summation applied to the Taylor's series of a function in regions where this series is divergent. The results obtained are those due to Borel and Mittag-Leffler and certain other investigators who have taken their methods as a point of departure. The various theories are unified by basing the whole discussion on a double integral formula which is one of the important generalizations of the well known integral formula of Cauchy. By this synthetic form of treatment Professor Buhl has found it possible to include an unusually large number of interesting results in the fifty odd pages of the monograph.

C. N. MOORE

Lehrbuch der Ballistik. By C. Cranz. Berlin, Julius Springer, 1925. xiv+711 pp.

This is the fifth edition of the author's first volume on ballistics, and it is confined to exterior ballistics. It is dedicated with appropriate courtesy to the late General A. von Kersting, the founder and first director of the Military Academy (1903–1912) who, during the first two years of the World War, rendered valuable service to German military technique as President of the Artillery Proving Commission. Proper acknowledgements are made to Major Becker for his work on ballistic wind and ballistic density, and to Professor O. von Eberhard who was the first to prove by actual computations the possibility of a trajectory of more than 100 km. in range.

Whatever might have been in the mind of Dr. Cranz in selecting the title of the book, the importance of its contents may be best described to the American reader by giving it the title *A Compendium of Exterior Ballistics*. The reviewer knows of no other volume on this subject that contains so much information together with such a thorough and masterly treatment of the subject. There are 539 pages of text and 267 pages of bibliography, ballistic tables, and ballistic diagrams. The references to literature are carefully assified as having special bearing upon definite articles. The bibliography alone makes the book highly desirable for a library.

In the first part of the book the author discusses in detail the trajectorv in vacuo with numerous illustrations of the path of a projectile fired from an elevation and from an inclined plane. These are interesting and, of course, elementary. If the reviewer is at liberty to give his real impressions of the contents of a book, possibly I may be pardoned for saying that one of these examples struck me with surprise. The problem is as follows: "Is it possible to throw a stone from the top of the Great Pyramid out over the base." After reading that its altitude is 137.2 m. and that the side of its square base is 227.5 m. one remains somewhat at a loss to know what the problem really is until he meets the statement that the velocity of throwing from the hand is assumed to be 24m/s which, he explains parenthetically, is the mean of thirty tests made upon as many different persons. He does not state how these tests were conducted, nor does he describe the physical characteristics of the persons who did the throwing. He shows that it is possible to clear the pyramid with an initial velocity of about 20m/s provided that the stone is thrown at the proper angle and he concludes that it is possible to throw a stone clear of the pyramid with " a certain amount of skill." Merely to throw a stone 20 m/s requires very little skill. Certainly the thirty persons would be poor candidates for a baseball team if the average velocity with which they throw a stone is about 65 f/s. The baseball player, I should think, throws with a velocity of 150 f/s to 165 f/s. Skill in this case consists in throwing at 20 m/s from the very restricted area at the top of the pyramid.

He follows the usual course of deriving the differential equations of the trajectory and then presenting the different methods of integrating these equations and of constructing the trajectory graphically. A number of graphical processes are described beginning with that of Poncelet in 1848

and coming down to those developed during the World War and later. Two of these, one by Vahlen in 1918 and another by Brauer in 1918, call forth the comment by Dr. Cranz that it would be a worthwhile problem for the investigator in ballistics to substantiate the sufficiency and utility of both of these graphical processes. The Vahlen method depends upon vectorial addition; that of Brauer is based upon the hodograph.

The author's presentation of satellite, parabolic, and hyperbolic velocities is of more than usual interest and the figures that accompany this discussion are especially attractive. The effect of yaw is considered in detail both from the theoretical and practical standpoints. This is followed by the fundamentals in projectile design for all types of projectiles.

The subjects of vertical, near-vertical, and long-range firing are treated under separate heads. The first two of these, of course, apply to anti-aircraft fire. The third is of very great interest to all those who are interested in modern ballistics. This is the portion of the book for which Dr. Cranz gives credit to Professor Eberhard. Comments are made upon phenomena observed abroad in 1914 which have been observed in this country since that time. A projectile was fired under initial conditions which, according to Siaccian methods of computing the trajectory, would have given a range of 38 Km.; the observed range was 49 Km. The angle of elevation which gave the greatest range was in the neighborhood of 55°. In this country unusually long ranges, viewed from the standpoint of Siaccian ballistics, have been obtained by firing at high angles of elevation, the angle of elevation giving the greatest range being above 45°, and comparatively little difference in range for angles between 45° and 55° elevation. This is explained by the fact that the projectile encounters less resistance as it rises through lighter and lighter strata than if it had passed through only comparatively denser air as in lower-angle fire.

Probably a better illustration of the effect of the rotation of the earth upon a projectile might have been given than the one selected by the author. Certainly it is not usual to fire with a velocity of 5000 f/s (1600 m/s). Until more firing has been done at such high velocity and the results observed, his illustration is more in the nature of the prediction of the pure mathematician than that of the conservative military engineer. If this particular example is given to register surprise, others with smaller initial velocities should have been given.

The arrangement of the material and the clearness of presentation are such that the book might be made the basis of a course of lectures on ballistics. Such a course could be made very elementary and involve merely an intelligent use of the tables and formulas, but by delving into the more theoretical portions of the book, especially the parts relating to the rotating projectile, it may be made exceedingly difficult. The volume is not only a *compendium* of facts and methods relating to exterior ballistics, written by one whose pen has enriched the literature on the general subject of ballistics for more than thirty years and filled with data that could be assembled and classified only by a lifelong interest in the subject, but it is also the product of a high type of thorough and mature scholarship.

J. E. ROWE

NOTES

Professor J. H. M. Wedderburn, of Princeton University, has been appointed an associate editor of this Bulletin. Professors R. W. Brink, of the University of Minnesota, and J. D. Tamarkin, of Dartmouth College, have been appointed associate editors of the Transactions of this Society.

The concluding number of volume 28 of the Transactions of this Society (October, 1926) contains the following papers: *A boundary value problem for a system of ordinary linear differential equations of the first order*, by G. A. Bliss; *On the theory of integral equations with discontinuous kernels*, by R. E. Langer; *The figuratrix in the calculus of variations*, by P. R. Rider; *On the existence of fields in boolean algebras*, by B. A. Bernstein; *A symmetric displacement of a vector*, by J. M. Thomas; *Tensors determined by a hypersurface in Riemann space*, by H. Levy; *A comparison of the series of Fourier and Birkhoff*, by M. H. Stone; *Analytic approximations to topological transformations*, by P. Franklin and N. Wiener.

The concluding number of volume 48 of the American Journal of Mathematics (October, 1926) contains: *Regular maps on an anchor ring*, by H. R. Brahana; *Determination of the type of the tricrunodal quartic by means of its invariants*, by L. T. Moore; *Subgroups of index p^2 contained in a group of order p^m*, by G. A. Miller; *Mapping by means of linear systems of curves invariant under Cremona involutions*, by T. L. Bennett; *Algebraic and transcendental equations connected with the form of stream lines*, by H. Bateman.

The Journal für die reine und angewandte Mathematik, founded in 1826 by August Leopold Crelle, is celebrating its centenary by the publication of a special Jubiläumsband.

On the occasion of the two hundredth anniversary of the death of Sir Isaac Newton, which occurs on March 20, 1927, a meeting will be held under the auspices of the (British) Mathematical Association at Grantham, near Newton's birthplace. Addresses will be delivered by Sir J. J. Thomson and others, on various aspects of his work.

Dr. Horace Lamb has been elected president of the Cambridge Philosophical Society.

Among the officers elected at the Philadelphia meeting of the American Association for the Advancement of Science are the following: Professor A. A. Noyes, of the California Institute of Technology, president; Professor Dunham Jackson, of the University of Minnesota, vice-president of Section A (mathematics); Professor A. H. Compton, of the University of Chicago, vice-president of Section B (physics); Dr. W. S. Adams, of Mount Wilson Observatory, vice-president of Section D (astronomy). Professor L. E. Dickson was elected a member of the Council, and Professor Oswald Veblen a member of the Committee on Grants.

Professor K. T. Compton, of Princeton University, has been elected president of the American Physical Society.

Nobel prizes in physics have been awarded as follows: for 1925, jointly to Professors J. Franck, of Göttingen, and G. Hertz, of Halle; for 1926, to Professor Jean Perrin, of the Sorbonne.

The prize of the American Association for the Advancement of Science for a paper read at its annual meeting of 1926, was awarded to Professor G. D. Birkhoff for his address as retiring president of the American Mathematical Society, delivered at a joint session of this Society with Section A of the Association. The address, entitled *A mathematical critique of some physical theories*, will appear in the next issue of this Bulletin.

Dr. Wolfgang Krull, of the University of Freiburg i. B., has been promoted to an associate professorship of mathematics.

Professor William H. Bussey has resigned the chairmanship of the Department of Mathematics at the University of Minnesota, in the College of Science, Literature and the Arts, so that he may devote more time to his duties as assistant dean for the junior college, and as editor-in-chief of the American Mathematical Monthly. Professor William L. Hart has been appointed chairman of the Department of Mathematics.

Dr. Selig Brodetsky, professor of applied mathematics at the University of Leeds, has arrived in the United States to make a tour of the country in the interest of the University of Jerusalem, of which he is one of the governors. He will lecture on *The Einstein theory of relativity* at Clark University on January 10; later he will give a series of lectures at Harvard University.

Associate Professor M. G. Gaba, of the University of Nebraska, has been promoted to a full professorship of mathematics.

Dr. Mayme I. Logsdon, of the University of Chicago, has been promoted to an assistant professorship of mathematics.

Assistant Professor F. S. Nowlan, of the University of Manitoba, has been appointed professor of mathematics at the University of British Columbia.

Professor E. Blaschke, of the Vienna Technical School, died October 30, 1926, at the age of seventy.

Professor Alexander Tschuprow, formerly of the Technical Institute at Petrograd, died April 19, 1926, at the age of fifty-two. He was known for his work in mathematical statistics.

Professor Louis Siff, of Louisville University, died December 25, 1926, at the age of fifty-seven.

Professor J. H. Tudor, of Pennsylvania State College, died June 7, 1926, at the age of sixty-nine. Professor Tudor was a member of this Society.

NEW PUBLICATIONS

PART I. PURE MATHEMATICS

BOULIGAND (G.) et RABATÉ (G.). Initiation aux méthodes vectorielles et aux applications géométriques de l'analyse. Paris, Vuibert, 1926. 2+215 pp.

CRUMLEY (T.). Logic, deductive and inductive. New York, Macmillan, 1926. 442 pp.

DURELL (C. V.). Projective Geometry. London, Macmillan, 1926. 10+222 pp.

EUCLID. See HEATH (T. L.).

FANO (G.). Lezioni di geometria descrittiva. Torino, Paravia, 1926. 19+461 pp.

GONSETH (F.). Les fondements des mathématiques. De la géométrie d'Euclid à la relativité générale et à l'intuitionisme. Paris, Blanchard, 1926. 16+243 pp.

VAN HAAFTEN (—.). Reziproken aller ganzen zahlen van 1 bis 10000. Aufgabe F von Noordhoff's Tafeln. Groningen, Noordhoff, 1926. 23+50 pp.

HADAMARD (J.) et MANDELBROJT (—.). La série de Taylor et son prolongement analytique. 2e édition, revue et mise au courant des progrès récents. (Scientia, No. 41.) Paris, Gauthier-Villars, 1926. 87 pp.

HASSE (H.). Höhere Algebra. I: Lineare Gleichungen. (Sammlung Göschen.) Berlin, de Gruyter, 1926. 160 pp.

HEATH (T. L.). The thirteen books of Euclid's elements translated from the text of Heiberg with introduction and commentary. 2d edition, revised with additions. 3 volumes. Cambridge, University Press, 1926. 432+436+546 pp.

HENDERSON (J.). Bibliotheca tabularum mathematicarum; being a descriptive catalogue of mathematical tables. Part 1: Logarithmic tables. (Tracts for Computers, No. 13.) Cambridge, University Press, 1926. 6+208 pp.

HOPF (E.). Ueber die Zusammenhänge zwischen gewissen höheren Differenzen-Quotienten reeller Funktionen einer reeller Variablen und deren Differenzierbarkeitseigenschaften. (Dissertation.) Berlin, Norddeutsche Buchdruckerei, 1926.

KRAITCHIK (M.). Théorie des nombres. Tome II: Analyse indéterminée du second degré et factorisation. Paris, Gauthier-Villars, 1926. 4+251 pp.

LAMOUCHE (A.). Le principe de simplicité dans les sciences mathématiques et dans les sciences physiques. Brest, La Dépêche, 1926.

LECAT (M.). Coup d'œil sur la théorie de la multiplication des déterminants supérieurs dans son état actuel. Louvain, Ceuterick, 1926. 40 pp.

LITTLEWOOD (J. E.). The elements of the theory of real functions. 2d (revised) edition. Cambridge, Heffer, 1926. 7+60 pp.

LÖTZBEYER (P.). Vierstellige Tafeln zum logarithmischen und natürlichen Rechnen. Ausgabe B. Leipzig, Teubner, 1926.

MAIR (D. B.). Fourfold geometry. Being the elementary geometry of the four-dimensional world. London, Methuen, 1926. 8+183 pp.

MANDELBROJT (—.). See HADAMARD (J.).

MUIKA (I.). Das Fermatsche Theorem. Bukarest, F. Göhl Söhne, 1926. 3 pp.

NEUGEBAUER (O.). Die Grundlagen der ägyptischen Bruchrechnung. Berlin, Springer, 1926. 46 pp.

PARAF-JAVAL (—.). Géométrie à toutes dimensions. Principes de géométric physique. Paris, Groupe d'Etudes Scientifiques, 1926.

RABATÉ (G.). See BOULIGAND (G.).

SAINT-LAGUË (A.). Les réseaux (ou graphes). (Mémorial des Sciences Mathématiques, No. 18.) Paris, Gauthier-Villars, 1926.

WOLF (A.). Excercises in logic and scientific methods. New edition, revised and enlarged. London, Allen and Unwin, 1926. 125 pp.

WOODS (F. S.). Advanced calculus. Boston, Ginn, 1926. 9+397 pp.

PART II. APPLIED MATHEMATICS

ARSOVITCH (R.). Pascal et l'expérience du Puy-de-Dome. Montpelier Imprimerie Firmin et Montane, 1925. 76 pp.

BARKHAUSEN (H.). Elektronen-Röhren. Band 2. Leipzig, Hirzel, 1925. 6+168 pp.

BEAVER (J. L.). Elements of alternating currents and alternating current apparatus. London, Longmans, 1926. 370 pp.

BETZ (A.). Wind-Energie und ihre Ausnutzung durch Windmühlen. Göttingen, Vandenhoeck und Ruprecht, 1926. 70 pp. +4 tables.

BLEICH (F.). Die Berechnung statisch unbestimmter Tragwerke nach der Methode des Viermomentensatzes. Berlin, Springer, 1925. 5+220 pp.

BREED (C. B.) and HOSMER (G. L.). The principles and practice of surveying. 3d edition, revised. Volume 2. London, Chapman and Hall, 1926. 507 pp.

BRUNET (P.). Les physiciens hollandais et la méthode expérimentale en France au XVIIIe siècle. Paris, A. Blanchard, 1926. 153 pp.

CARSTANGEN (M.). See KOCH (A.).

CORPS (—.). Les origines de l'erreur relativiste. La nouvelle expérience de M. Michelson. Paris, Les Presses Universitaires de France, 1926. 20 pp.

CORRAL (J. I.). Nueva solución del problema de Lord Kelvin sobre ecuaciones de coeficientes reales. Madrid, Imprenta Clasica Española, 1926.

CRANDALL (I. B.). Theory of vibrating systems and sound. New York, Van Nostrand, 1926. 10+272 pp.

CREHORE (A. C.). The progress of atomic theory. London, Taylor and Francis, 1926.

EYRAUD (H.). Les équations de la dynamique de l'éther, avec une note sur la technique du repérage de l'espace et du temps dans ses rapports avec la relativité. Paris, A. Blanchard, 1926. 68 pp.

FORESTIER (A.). L'énergie rayonnante. 2e édition, augmentée. Paris, A. Blanchard, 1926. 76 pp.

FRANCK (J.) und JORDAN (P.). Anregung von Quantensprüngen durch Stösse. Berlin, Springer, 1926. 8+312 pp.

GARRETT (H. E.). Statistics in psychology and education. New York, Longmans, 1926. 14+317 pp.

GEIGER (H.) und SCHEEL (K.), herausgegeben von. Handbuch der Physik. Band 1: Geschichte der Physik. Vorlesungstechnik. Redigiert von K. Scheel. Band 2: Elementare Einheiten und ihre Messung. Redigiert von K. Scheel. Band 9: Theorien der Wärme. Redigiert von F. Henning. Band 17: Elektrotechnik. Redigiert von W. Westphal. Band 23: Quanten. Redigiert von H. Geiger. Berlin, Springer, 1926. 6+404+8+522+8+616+399+9+782 pp.

GIBSON (A. H.). Hydraulics and its applications. 3d edition. London, Constable, 1925. 15+801 pp.

GLAUERT (H.). The elements of aerofoil and airscrew theory. Cambridge, University Press, 1926. 8+228 pp.

HAAS (A.). See WIEN (W.).

HARMS (F.). See WIEN (W.).

HENNING (F.). See GEIGER (H.).

HOSMER (G. L.). See BREED (C. B.).

ILBERG (W.). Drahtlose Telegraphie und Telephonie in ihren physikalischen Grundlagen. Leipzig, Teubner, 1925.

JOLY (G.). Nouvelle étude physique de l'expérience de MM. Michelson et Morley qui sert de base à la théorie de la relativité. Paris, Editions Spes, 1926. 8 pp.

JORDAN (P.). See FRANCK (J.).

VON JÜPTNER (H.). Gas, Dampf und Flüssigkeit. Leipzig, O. Spammer, 1925.

KAFKA (H.). Die ebene Vektorrechnung und ihre Anwendungen in der Wechselstromtechnik. Teil 1: Grundlagen. Leipzig, Teubner, 1926. 8+132 pp.

KAPP (G.). Transformers for single and multiphase currents. A treatise on their theory, construction and use. 3d edition, revised by R. O. Kapp. London, Pitman, 1925.

KAPP (R. O.). See KAPP (G.).

KOCH (A.). Von der Bewegung des Wassers und den dabei auftretenden Kräften, nach Arbeiten von Staatsrat. Herausgegeben von M. Carstangen. Berlin, Springer, 1926.

LEPAPE (A.). See MILLIKAN (R. A.).

LEWITT (E. H.). The rigid airship. A treatise on the design and performance. London, Pitman, 1925. 10+283 pp.

LIMBURG (H.). Onderzoekingen over emulsies. (Dissertation, Delft.) Delft, 1924. 126 pp.

LONEY (S. L.). Solutions of the examples in a treatise on dynamics of a particle and of rigid bodies. Cambridge, University Press, 1926. 6+240 pp..

MANEY (G. A.). See PARCEL (J. I.).

MARCUS (H.). Die vereinfachte Berechnung biegsamer Platten. Berlin, Springer, 1925. 92 pp.

METZ (A.). Les nouvelles théories scientifiques et leurs adversaires. La relativité. 15e édition. Paris, Chiron, 1926. 270 pp.

MILLIKAN (R. A.). L'électron. Traduit sur la 2e édition américaine par A. Lepape. Paris, Alcan, 1926. 346 pp.

O'ROURKE (C. E.). See URQUHART (L. C.).

PARCEL (J. I.) and MANEY (G. A.). An elementary treatise on statically indeterminate stresses. New York, Wiley, 1926. 368 pp.

PAYNE (C. H.). Stellar atmospheres. Cambridge, Harvard University Press, 1925. 9+215 pp.

PEAKE (E. G.). An academic study of some money market and other statistics. 2d edition. London, King, 1926. 1+105 pp.

PERTSCH (J. G.). Electrical engineering problems. Part 1. New York, McGraw-Hill, 1925. 8+213 pp.

PETITET (A.). See ROUSSELET (L.).

PÖSCHL (T.), herausgegeben von. Berechnung von Behältern nach neueren analytischen und graphischen Methoden. 2te Auflage. Berlin, Springer, 1926.

POPOFF (K.). Les méthodes d'intégration de Poincaré et le problème général de la balistique extérieure. Paris, Gauthier-Villars, 1925.

PRITCHARD (J. L.). Broadcast reception in theory and practice. London, Chapman and Hall, 1926. 11+259 pp.

ROUSSELET (L.) et PETITET (A.). Stabilité des constructions usuelles. Calculs algébriques et graphiques des efforts dans l'ossature principale. 2e édition, revue, corrigée et augmentée. Paris, Dunod, 1926. 6+329 pp.

SCHEEL (K.). See GEIGER (H.).

SCHREIBER (P.). Wärmemechanik wasserhaltiger Gasgemische mit den Hilfsmitteln der Flächennomographie bearbeitet. Braunschweig, Vieweg, 1925. 8+195 pp.

SPREEN (W.). Formeln und Tabellen aus dem Gebiete der Funktechnik. Berlin, Springer, 1925. 80 pp.

STUYVAERT (M.). Statique et dynamique. 2e édition. Gand, Van Ryssel-berg et Rombaut, 1926. 208 pp.

THOMAS (P.). Calcul des constructions continues à éléments droits. Application aux arcs hyperstatiques. Paris, Béranger, 1925.

URQUHART (L. C.) and O'ROURKE (C. E.). Steel structures. Stresses in simple structures. New York, McGraw-Hill, 1926. 9+278 pp.

WESTPHAL (W.). See GEIGER (H.).

WIEN (W.) und HARMS (F.). Handbuch der Experimentalphysik. Band 2: Mechanik der Massenpunkte und der starren Körper, von Arthur Haas. Leipzig, Akademische Verlagsgesellschaft, 1926. 14+355 pp.

Mémorial des Sciences Mathématiques

Edited by HENRI VILLAT

Under the patronage of la Société mathématique de France, l'Académie des Sciences de Paris, and the scientific academies of Servia, Belgium, Bucarest, Poland, Ukraine, Spain, Prague, Rome (dei Lincei), and Sweden. See this BULLETIN, vol. 31, Nos. 5-6, p. 281.

The following seventeen monographs have appeared at 10 fr. each.

PAUL APPELL, *Sur une forme générale des équations de la dynamique.*
G. VALIRON, *Fonctions entières et fonctions méromorphes.*
PAUL APPEL, *Séries hypergéométriques de plusiers variables, polynomes d'Hermite et autres fonctions sphériques de l'hyperespace.*
M. D'OCAGNE, *Esquisse d'ensemble de la nomographie.*
P. LÉVY, *Analyse fonctionnelle.*
E. GOURSAT, *Lo problème de Bäcklund.*
A. BUHL, *Séries analytiques. Sommabilité.*
TH. DE DONDER, *Introduction à la gravifique einsteinienne.*
E. CARTAN, *La géométrie des espaces de Riemann.*
P. HUMBERT, *Fonctions de Lamé et fonctions de Mathieu.*
G. BOULIGAND, *Fonctions harmoniques. Principes de Picard et de Dirichlet.*
R. GOSSE, *La méthode de Darboux pour les équations* $s=f(x, y, z, p, q)$.
A. VÉRONNET, *Figures d'équilibre et cosmogonie.*
TH. DE DONDER, *Théorie des champs gravifiques.*
S. ZAREMBA, *La logique des mathématiques.*
A. BUHL, *Formules Stokiennes.*
G. VALIRON, *Théorie générale des séries de Dirichlet.*

Many others are in press or in preparation.

For all information, address M. Villat, l'Université de Strasbourg, or

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OFFICIAL COMMUNICATIONS

Meetings of the Society have been fixed at the following times and places:

STANFORD UNIVERSITY, April 2, 1927. (The San Francisco Section.)

Abstracts must be in the hands of the Secretary of the Section, Professor B. A. Bernstein, not later than March 12, 1927.

CHICAGO, April 15-16, 1927.

Abstracts must be in the hands of the Assistant Secretary of the Society, Professor Arnold Dresden, 2114 Vilas St., Madison, Wis., not later than March 25, 1927. At the request of the program committee, Professor E. W. Chittenden will deliver an address on *Some phases of general topology.*

NEW YORK CITY, May 7, 1927

Abstracts must be in the hands of the Secretary of the Society, Professor R. G. D. Richardson, 501 West 116th St., New York City, not later than April 16.

VANCOUVER, BRITISH COLUMBIA, June 18, 1927. (The San Francisco Section.)

Abstracts must be in the hands of the Secretary of the Section, Professor B. A. Bernstein, University of California, Berkeley, not later than May 27.

MADISON, WISCONSIN, SUMMER MEETING AND COLLOQUIUM. September 6-9, 1927.

Abstracts must be in the hands of the Secretary of the Society, Professor R. G. D. Richardson, 501 West 116th St., New York City, not later than August 8. Series of colloquium lectures will be delivered by Professor E. T. Bell on *Algebraic arithmetic,* and by Professor Anna Pell-Wheeler on *The theory of quadratic forms in infinitely many variables, and applications.*

R. G. D. RICHARDSON,
Secretary of the Society.

Articles for insertion in the BULLETIN should be addressed to E. R. HEDRICK, Editor of the BULLETIN, University of California, Los Angeles, Calif. Reviews should be sent to W. R. LONGLEY, Yale University, New Haven, Conn. Notes should be sent to H. W. KUHN, Ohio State University, Columbus, Ohio.

Subscriptions to the BULLETIN, orders for back numbers, and inquiries in regard to non-delivery of current numbers should be addressed to The American Mathematical Society, Menasha, Wis., or 501 West 116th Street, New York.

Advertising space is available in the BULLETIN at $16 per page, $9 per half page, $5 per quarter page, per issue. Address correspondence regarding advertising to H. L. RIETZ, University of Iowa, Iowa City, Iowa.

Changes of address or members should be communicated at once to the Secretary of the Society, R. G. D. RICHARDSON, 501 West 116th Street, New York.

The initiation fees ($5.00) and the annual dues ($6.00) of members of the Society are payable to the Treasurer of the Society, W. B. FITE, 501 West 116th Street, New York.

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